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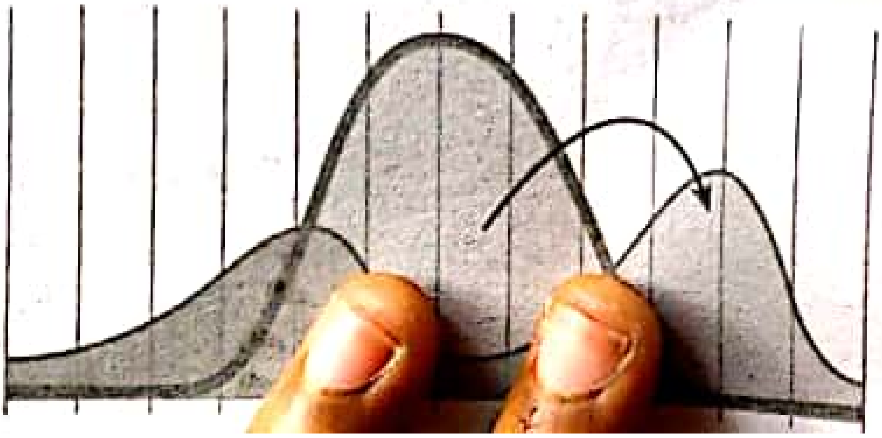

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1 CHAPTER

Basic Ideas of Ordinary Differential Equation

Introduction

An equation which contains at least one differential coefficient, is known as differential equation. In any differential equation, there is at least one independent variable with respect to whom differentiation is done and at least one dependent variable. Among various types of differential equation the common types of differential equation are ordinary differential equation, partial differential equation, delay differential equation, fuzzy differential equation, stochastic differential equation, functional differential equation, fractional differential equation etc. Here we are discussing only ordinary differential equation. An ordinary differential equation is an equation which contains ordinary derivatives.

- **Example :** (i) $y\frac{dy}{dx} + x = 0$ (ii) $\frac{dy}{dx} + x = 0$

  (iii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2x = 4$ etc.

An ordinary differential equation contains always differential coefficients with one dependent variable and one independent variable.

1.1 Order and Degree of Ordinary Differential Equation

The highest order derivative present in the equation is the **order** of the differential equation. For example the order of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + 2\frac{dy}{dx} + x^2 = 0$ is 2 where as the order of the differential equation $\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + 2y = x^2$ is 3. The power of the highest order derivative used in the rationalized form in a differential equation is its **degree.** For example, the degree of $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is 1, $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + x^2 + y^2 = 0$ is 2, $\sqrt{\frac{dy}{dx}} = 3y + 2x$ is 1.

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Note

Consider the following differential equations

(i) $\cos\left(\frac{dy}{dx}\right) + 2\left(\frac{dy}{dx}\right)^2 + y = 0$ (ii) $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$ (iii) $\frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} + 2y = 0$

The highest order derivative present in the above three differential equations are 1, 4 and 2 respectively. So the order of each of the differential equations are defined but they cannot be expressed as a polynomial in differential coefficients. So the degree of each of the above three differential equations are not defined.

1.2 Linear and Non-linear Ordinary Differential Equation

An n-order differential equation of the following form

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = f(x) \qquad \ldots (1)$$

Differential equation (1) is a linear differential equation of order n if a_k ($k = 0, 1, \ldots, n$) and $f(x)$ are either constant or function of x only with $a_0 \neq 0$.

However a differential equation in any other form is known as non-linear differential equation.

In particular the general form of a first order and second order linear ordinary differential equation are respectively $\frac{dy}{dx} + a_1y = f_1(x)$ and $\frac{d^2y}{dx^2} + b_1\frac{dy}{dx} + b_2y = f_2(x)$

where a_1, b_1, b_2, $f_1(x)$ and $f_2(x)$ all are either constant or function of x only.

1.3 Solution of a Differential Equation

The general form of a n-th order differential equation is $\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \quad \ldots (2)$

A function $y = g(x)$ is said to be a solution of above differential equation if it satisfies the above equation, *i.e.*, if we have, $\phi(x, g(x), g'(x), g''(x), \ldots, g^{(n)}(x)) = 0$.

For example the function $y = e^x$ is a solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$. Further it is observed that if c be any arbitrary constant, then ce^x also satisfy the equation $\frac{d^2y}{dx^2} - y = 0$ and so $y = ce^x$ is also a solution, in fact it is more general solution than $y = e^x$. However, corresponds to an n-degree differential equation, the differential equation possesses n independent solutions. The general solution of a differential equation will be then a linear combination of those n independent solutions. For example, like $y = e^x$, the function $y = e^{-x}$ (which is independent to e^x) is another solution of the equation $\frac{d^2y}{dx^2} - y = 0$. Therefore the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$ is $y = ce^x + de^{-x}$. where c and d are two arbitrary constants.

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Note

An n-order differential equation has exactly n linearly independent solution and n-arbitrary constants. Similarly corresponds to a first order differential equation, there is only one solution and one arbitrary constant and corresponding to a second order equation, there are two independent solutions along with two arbitrary constants.

With the help of the given initial or boundary conditions, when the suitable values of the arbitrary constants are determined then the reduced form of solution is known as particular solution.

- **Example :** The independent solutions of $\frac{d^2y}{dx^2} + y = 0$ are $y = \sin x$ and $y = \cos x$ so that the general solution is $y = A\cos x + B\sin x$, where A and B are two arbitrary constants. Now if we impose the boundary conditions $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 1$, then we have $A = 0$ and $B = 1$. Thus the particular solution will be $y = \sin x$.

Note

Every particular solution can be obtained from substituting suitable value(s) at the arbitrary constant(s) of the general solution of a differential equation.

1.4 Formation of Differential Equation : Application to Geometry

Let us consider the equation of the family of straight lines which makes an angle $\frac{\pi}{4}$ to the positive direction of abscissa. Then the equation of the family of straight lines is $y = x + c$, c being an arbitrary constant. The corresponding differential equation of that family of straight lines is $\frac{dy}{dx} = 1$. Hence the family of the straight lines $y = x + c$ can be represented by the differential equation $\frac{dy}{dx} = 1$.

Thus corresponds to every family of conic, there is an equivalent differential equation, which also represents the same conic, can be obtained by eliminating the arbitrary constant(s) presented in the expression of the family of conic. Further it is also to be noted that number of arbitrary constants of the expression of a conic in Cartesian or Polar coordinates will be same as the order of the corresponding differential equation. For example if we consider a family of curves of the form $y^2 = bx$, where b is a arbitrary parameter, then the corresponding differential equation form of the curve $y^2 = bx$ is $y^2 = 2y\left(\frac{dy}{dx}\right)x$ i.e., $y - 2x\frac{dy}{dx} = 0$.

Since there is only one parameter, so order of the differential equation is 1.

Similar to geometrical problem, almost every problem related to daily life can be incorporated with the help of differential equation.

1.5 Application of Differential Equation : Orthogonal Trajectories

Suppose there is a family of curves $F(x, y, \alpha) = 0$ where α is a parameter. If we are able to find another family of curve G which intersect each member of F orthogonally *i.e.*, the

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angle between the tangents at the itersecting points of the two families of curves F and G is $\frac{\pi}{2}$ then the family of curve G is said to be orthogonal trajectories of the family $F(x, y, \alpha) = 0$.

We note that for orthogonal trajectories, the tangents at the intersecting point between those two family of curves are orthogonal, i.e. the product of the slope at that point will be –1.

Therefore if the given curve and orthogonal trajectories meets at (x, y) and slope of the given family of curve is $\frac{dy}{dx}$, then slope of orthogonal trajectories will be $-\frac{dx}{dy}$. Note that if F and G are identical forms then they are said to be self-orthogonal.

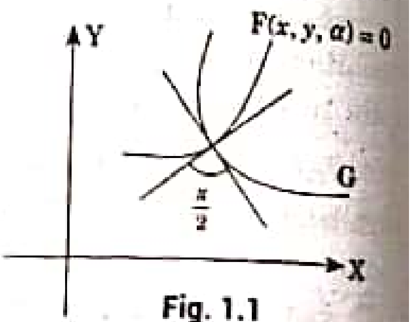

Fig. 1.1

1.6 Working rule to find Orthogonal Trajectories of $F(x, y, \alpha) = 0$

- **Step-1 :** Eliminate α between $F(x, y, \alpha) = 0$ and $\frac{dy}{dx} = F_1(x, y, \alpha)$, where $\frac{dy}{dx} = F_1(x, y, \alpha)$ can be obtained by differentiating $F(x, y, \alpha) = 0$ with respect to x. Let the family of curve after eliminating α becomes $\phi\left(x, y, \frac{dy}{dx}\right) = 0$.
- **Step-2 :** Replace $\frac{dy}{dx}$ by $\left(-\frac{dx}{dy}\right)$ so that the differential equation of family of orthogonal curves become $\phi\left(x, y, -\frac{dx}{dy}\right) = 0$.
- **Step-3 :** Solve the differential equation $\phi\left(x, y, -\frac{dx}{dy}\right) = 0$, to obtain the family of orthogonal trajectories.

1.7 Working rule to find Orthogonal Trajectories in Polar Coordinates

Let the family of given curve in $F(r, \theta, \alpha) = 0$, α is a parameter and eliminating α, given family reduces to the form $\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0$. Then replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, the equation $\phi\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$ represents the differential equation for the family of orthogonal trajectories and an integration, we get the required family of orthogonal trajectories.

Note

To find a trajectory, instead of orthogonal trajectory in Cartesian coordinates, we have to replace $\frac{dy}{dx}$ by $\left(\frac{\frac{dy}{dx} - \tan\theta}{1 + \left(\frac{dy}{dx}\right)\tan\theta}\right)$, where '$\theta$' be the angle between given curve and its trajectory.

In particular for $\theta = \frac{\pi}{4}$, then to find trajectories, required replacement will be $\left(\frac{\frac{dy}{dx} - 1}{1 + \frac{dy}{dx}}\right)$.

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Illustrative Examples

1. **Find the order and degree of the differential equation** [CP 2013]

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^5 + 9y = 0$$

Solution Here the given differential equation is $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^5 + 9y = 0$ ... (1)

The highest order of differential term involving in (1) is 2 and so the given differential equation is of second order.

Further the differential equation is in rationalized form and the power of highest order differential term in 1. Therefore the given differential equation is of degree 1.

2. **Find the order and degree of the differential equation** $\frac{d^2y}{dx^2} = \sqrt[3]{1-\left(\frac{dy}{dx}\right)^4}$.

Solution Here the given differential equation is $\frac{d^2y}{dx^2} = \sqrt[3]{1-\left(\frac{dy}{dx}\right)^4}$

or, $\left(\frac{d^2y}{dx^2}\right)^3 = 1 - \left(\frac{dy}{dx}\right)^4$ ... (1)

Since the highest order of differential term involving (1) is 2, so the given differential equation is of second order.

Further the differential euqation is in rationalized form and the power of highest order differential term is 3.

Therefore the given differential equation is of degree 3. So order = 2 and degree = 3.

3. **Find the order of the differential equation of all parabolas whose axis of symmetry is parallel to x-axis.**

Solution The general equation of all parabolas whose axis of symmetry is parallel to x-axis is $x = ay^2 + by + c$, where a, b, c are three arbitrary constants. So the order of the differential equation is 3.

4. **Find the order of the differential equations of all conics whose centre is at the origin.**

Solution We have the general equation of all conics whose centre lie at the origin is $ax^2 + 2hxy + by^2 = 1$ where a, h, b are three arbitrary constants.

So the order of the differential equation is 3.

5. **Find the degree of the differential equation satisfying $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$.**

Solution Here the given equation is $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$.

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Putting $x = \sin\theta$ and $y = \sin\phi$, we get $\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

or, $2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 2a\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}$

or, $\cot\frac{\theta-\phi}{2} = a$ or $\theta - \phi = 2\cot^{-1}a$

or, $\sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$ (constant)

Differentiating with respect to x, we get $\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$

i.e., $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

So, the degree of the differential equation is 1.

6. **Find the differential equation of the curve $y = ae^x + be^{-x}$, where a, b are parameters.** [CP 2012]

Solution Here the given equation of curve is : $y = ae^x + be^{-x}$, ... (1)

Since the curve (1) contains two parameters a and b, therefore, the required differential equation will be of second order.

Differentiating (1) with respect to x two times, we have, $\frac{dy}{dx} = ae^x - be^{-x}$

$\frac{d^2y}{dx^2} = ae^x + be^{-x} = y$ *i.e.*, $\frac{d^2y}{dx^2} - y = 0$, which is the required differential equation.

7. **Find the differential equation of the family of curves $xy = Ae^x + Be^{-x} + x^2$, where A and B are parameters.** [CP 2013]

Solution Here the given equation of curve is $xy = Ae^x + Be^{-x} + x^2$ ... (1)

Since the conic (1) has two parameters A and B, therefore, the corresponding differential equation will be of order two.

Differentiating (1) with respect to x twice, we have,

$$x\frac{dy}{dx} + y = Ae^x - Be^{-x} + 2x$$

and $$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = Ae^x + Be^{-x} + 2$$

i.e., $$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (xy - x^2) + 2 \quad \text{[from (1)]}$$

i.e., $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 2 - x^2$, which is the required differential equation.

8. **Show that the differential equation of all parabolas having their axes parallel to the y-axis is $\frac{d^3y}{dx^3} = 0$** [CP 1994, 2011]

Solution The equation of the family of all parabolas whose axes are parallel to the y-axis in Cartesian coordinates is

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$y = ax^2 + bx + c$ ... (1)

where a, b, c are three parameters.

Since the conic (1) contains three parameter, therefore the required differential equation will be of order 3.

Differentiating (1) with respect to x thrice, we have $\frac{dy}{dx} = 2ax + b$

$\frac{d^2y}{dx^2} = 2a$ and $\frac{d^3y}{dx^3} = 0$, which is the required equation **[Proved]**

9. **Find the differential equation of all circle whose centre is at origin.** [CP 2010]

Solution Equation of all circle whose centre is at origin is $x^2 + y^2 = r^2$ ... (1)

where r is a parameter.

Differentiating (1) with respect to x once we have

$2x + 2y\frac{dy}{dx} = 0$ *i.e.*, $y\frac{dy}{dx} + x = 0$, which is the required differential equation.

10. **Find the differential equation of the curve $y = a\cos 2x + b\sin 2x$, where a and b are two parameters.** [CP 2016]

Solution Here the equation of given curve is $y = a\cos 2x + b\sin 2x$ ... (1)

Since (1) contains two parameters, therefore required differential equation would be a second order differential equation.

Differentiating (1) with respect to x twice we get $\frac{dy}{dx} = -2a\sin 2x + 2b\cos 2x$

$\frac{d^2y}{dx^2} = -4a\cos 2x - 4b\sin 2x = -4(a\cos 2x + b\sin 2x) = -4y$

i.e., $\frac{d^2y}{dx^2} + 4y = 0$, which is the required differential equation.

11. **Find the differential equation of the parabolas represented by $y = x^2 + 2cx$, c being an arbitrary constant.** [CP 2014]

Solution Here the equation of given curve is $y = x^2 + 2cx$ ... (1)

where c is an arbitrary constant.

Since there is only one arbitrary constant, therefore the required differential equation will be of order one.

Differentiating (1) with respect to x once we have

$\frac{dy}{dx} = 2x + 2c$ *i.e.*, $2c = \left(\frac{dy}{dx} - 2x\right)$ ... (2)

Eliminating c between (1) and (2) we have

$y = x^2 + x\left(\frac{dy}{dx} - 2x\right) = x\frac{dy}{dx} - x^2$

i.e., $x\frac{dy}{dx} - x^2 - y = 0$, which is the required differential equation.

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Alternative Method

Here the equation of given curve is $y = x^2 + 2cx$ i.e., $\frac{y}{x} = x + 2c$

Now differentiating with respect to x, we get $\frac{x\frac{dy}{dx} - y}{x^2} = 1$

i.e., $x\frac{dy}{dx} - x^2 - y = 0$ which is the required differential equation.

12. **Find the differential equation of the family of circles having fixed radius *r*.** [CP 1990, 1997, 2015]

Solution The equation of the family of circles having fixed radius r and centre at (a, b) is

$(x - a)^2 + (y - b)^2 = r^2$ ... (1)

where a, b are two arbitrary constants.

Since there are two arbitrary constants in (1), so the required differential equation will be of second order.

Differentiating (1) with respect to x twice, we have

$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0 \text{ i.e., } (x - a) + (y - b)\frac{dy}{dx} = 0 \quad \text{... (2)}$$

$$(y - b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0 \text{ i.e., } y - b = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

Substituting $y - b = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$ in (2), we have $x - a = \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}$

Substituting the values $(x - a)$ and $(y - b)$ in (1), we have

$$\left(\frac{dy}{dx}\right)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2 + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^2 = r^2\left(\frac{d^2y}{dx^2}\right)^2 \text{ i.e., } \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

i.e., $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = r\frac{d^2y}{dx^2}$, which is the required differential equation.

13. **Find the equation of the curve whose slope at any point (x, y) on it is xy and which passes through $(0, 1)$** [CP 2012, 2014]

Solution We know that the slope of a curve at the point (x, y) is $\left(\frac{dy}{dx}\right)_{(x,y)}$ and according to given condition $\left(\frac{dy}{dx}\right)_{(x, y)} = xy$

i.e., $\frac{dy}{y} = x\,dx$

on integrating we have $\log cy = x$, where c is an arbitrary constant,

i.e., $cy = e^x$ ... (1)

Again since the curve (1) passes through the point $(0, 1)$, then $c \cdot 1 = e^0$ given $c = 1$

Hence the equation of the required curve is $y = e^x$.

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14. **Find the differential equation of all non-vertical lines in a plane.**

Solution Since the general equation of all non-vertical lines in a plane is $ax + by = 1$ where $b \neq 0$, so it has two arbitrary constants.

Now differentiating with respect to x, we have $a + b\frac{dy}{dx} = 0$

Again differentiating with respect to x, we have $b\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$ [since $b \neq 0$]
which is the required differential equation.

15. **Find the differential equation representing all tangent to the parabola $y^2 = 2x$.**

Solution Since we have the equation of any tangent to the parabola $y^2 = 2x$ is

$y = mx + \frac{a}{m}$, where m is an arbitrary constant. ... (1)

Comparing the parabola $y^2 = 2x$ with $y^2 = 4ax$, we have

$4a = 2$ *i.e.*, $a = \frac{1}{2}$, So from (1), equation of tangent is $y = mx + \frac{1}{2m}$ ... (2)

Differentiating, we have $m = \frac{dy}{dx}$

Thus from (2), we have $y = x\frac{dy}{dx} + \frac{1}{2}\frac{1}{\frac{dy}{dx}}$ or, $2x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + 1 = 0$

Which is the required differential equation.

16. **Obtain the differential equation of all circles each of which touches the axis of x at the origin.** [CP 2013, 2017, CH 1985, VH 1988]

Solution Equation of the circles touching the x-axis at the origin is $x^2 + y^2 = 2ay$ ... (1)

a being a parameter or an arbitrary constant

Differentiating both sides of (1) with respect to x, we have

$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$ *i.e.*, $a = \frac{x + yy_1}{y_1}$, where $y_1 = \frac{dy}{dx}$

Substituting, $a = \frac{x + yy_1}{y_1}$ in (1), we have $x^2 + y^2 = 2y\left(\frac{x + yy_1}{y_1}\right)$ *i.e.*, $(x^2 - y^2)\frac{dy}{dx} = 2xy$

i.e., $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$, which is the required differential equation.

17. **Find the orthogonal trajectories of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, where a is a parameter.** [CP 2014, 2017]

Solution Here the given curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ ... (1)

Differentiating (1) with respect to x, we have

$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0$ *i.e.*, $x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\frac{dy}{dx} = 0$ *i.e.*, $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$

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Therefore the curve which meets orthogonally can be determined from the equation

$-\frac{dx}{dy} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$ i.e., $\frac{dy}{dx} = \left(\frac{x}{y}\right)^{\frac{1}{3}}$ i.e., $x^{\frac{1}{3}}dx = y^{\frac{1}{3}}dy$ and on integration we have

$\frac{3}{4}x^{\frac{4}{3}} + \frac{3}{4}c^{\frac{4}{3}} = \frac{3}{4}y^{\frac{4}{3}}$ (where c is an arbitrary constant)

i.e., $y^{\frac{4}{3}} = x^{\frac{4}{3}} + c^{\frac{4}{3}}$, which is the equation of required orthogonal trajectories.

18. **Find the orthogonal trajectories of the family of parabolas $y^2 = 4a(x+a)$, a being a parameter.** [CP 2012, 2015]

Solution Equation of the given curve is $y^2 = 4a(x + a)$ ... (1)

Differentiating (1) with respect to x, we have, $2y\frac{dy}{dx} = 4a$

i.e., $a = \frac{1}{2}(yp)$ where $p = \frac{dy}{dx}$

Substituting the value of a in (1), we have

$$y^2 = 2yp\left(x + \frac{1}{2}yp\right) = yp(2x + yp) \quad i.e., y = 2px + p^2y \qquad ... (2)$$

Now the orthogonal trajectories of the curve (1) can be obtained by solving the differential equation which can be obtained by substituting p by $\left(-\frac{1}{p}\right)$.

So, the differential equation for orthogonal trajectories becomes

$$y = -\frac{2x}{p} + \frac{1}{p^2}y \quad i.e., y = 2px + p^2y \qquad ... (3)$$

Clearly (3) is the same form of (2) and thus the solution of (3) will be same as form of the curve (1). Thus we take the solution of (3) as $y^2 = 4c(x + c)$, where c is an arbitrary constant.

Hence the orthogonal trajectories to the family of parabolas $y^2 = 4a(x + a)$ is $y^2 = 4c(x + c)$.

19. **Find the orthogonal trajectories of the family of curves $r^n = a^n\cos n\theta$, a being the parameter.** [CP 2014]

Solution Equation of the given curve is $r^n = a^n\cos n\theta$ ... (1)

where a is a parameter.

Differentiating (1) with respect to θ we have,

$$nr^{n-1}\frac{dr}{d\theta} = -na^n\sin n\theta \quad i.e., r^{n-1}\frac{dr}{d\theta} = -a^n\sin n\theta \qquad ... (2)$$

Eliminating a^n from (1) and (2), we have $\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$ ... (3)

Equation (3) becomes the differential equation of the given family of curve. Now to find the orthogonal family of curves we replace $\frac{dr}{d\theta}$ by $\left(-r^2\frac{d\theta}{dr}\right)$. Then the differential equation for orthogonal family of curves become $\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan n\theta$ i.e., $\frac{dr}{r} = \cot n\theta d\theta$

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Integrating we have, $\log c_1 r = \frac{1}{n}\log\sin n\theta$, where c_1 is an arbitrary constant.

i.e., $\log c_1^n r^n = \log\sin n\theta$ *i.e.*, $r^n = c^n \sin n\theta$ where $c^n = \frac{1}{c_1^n}$ is another arbitrary constant, which is the required equation of family of orthogonal trajectories.

20. **Find the orthogonal trajectories of the family of circles $x^2 + y^2 = 2ay$, a being a parameter.** [CP 2013, 2016]

Solution Here the equation of the given family of circles is $x^2 + y^2 = 2ay$ ... (1)

where a is a parameter.

Differentiating (1) with respect to x we have, $2x + 2yp = 2ap$, where $p = \frac{dy}{dx}$

i.e., $a = \frac{py + x}{p}$

Therefore the differential equation which represents the family of circle (1) can be obtained as follows

$$x^2 + y^2 = 2y\left(\frac{py + x}{p}\right) \quad i.e., \; p(x^2 + y^2) = 2y(py + x) \quad \text{... (2)}$$

To find the differential equation of orthogonal family of curves, we substitute p by $\left(-\frac{1}{p}\right)$ in (2) and then we have

$$-\frac{1}{p}(x^2 + y^2) = 2y\left(x - \frac{y}{p}\right) \quad i.e., \; 2pxy = y^2 - x^2$$

$$i.e., \; p = \frac{y^2 - x^2}{2xy} \quad i.e., \; \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \text{... (3)}$$

To solve (3), we substitute $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then (3) reduces to $v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$ *i.e.*, $x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = -\frac{v^2 + 1}{2v}$ *i.e.*, $\frac{dx}{x} + \frac{2v}{v^2 + 1}dv = 0$

Integrating we have, $\log x + \log(v^2 + 1) = \log c$, c being an arbitrary constant.

i.e., $x\left(\frac{y^2}{x^2} + 1\right) = c$ *i.e.*, $x^2 + y^2 = cx$

which is the required equation of family of orthogonal trajectories.

21. **Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, λ being a parameter.** [CP 2011, 2013; CH 1992]

Solution Here the equation of confocal conics are $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ ... (1)

where λ is a parameter.

Differentiating (1) with respect to x, we have, $\frac{2x}{a^2 + \lambda} + \frac{2y}{b^2 + \lambda}\frac{dy}{dx} = 0$

i.e., $(x + yp)\lambda = -(b^2 x + a^2 yp)$ where $p = \frac{dy}{dx}$ *i.e.*, $\lambda = -\frac{b^2 x + a^2 yp}{x + py}$

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Now $a^2 + \lambda = a^2 - \frac{b^2x + a^2py}{x + py} = \frac{(a^2 - b^2)x}{x + py}$ and $b^2 + \lambda = b^2 - \frac{b^2x + a^2py}{x + py} = -\frac{(a^2 - b^2)p}{x + py}$

Thus the differential equation which represents the conic (1) is

$$\frac{x^2(x + py)}{(a^2 - b^2)x} - \frac{y^2(x + py)}{(a^2 - b^2)py} = 1 \quad i.e., \ (x + py)\left(x - \frac{y}{p}\right) = a^2 - b^2 \qquad \text{... (2)}$$

The orthogonal family of conics of (1) can be found by substituting $-\frac{1}{p}$ is place of p i (2). Thus the differential equation of orthogonal trajectories become :

$$\left(x - \frac{y}{p}\right)(x + py) = a^2 - b^2 \qquad \text{... (3)}$$

clearly the differential equation (3) and (2) are of same form. Therefore the solution of (3) can be written as $\frac{x^2}{a^2 + c} + \frac{y^2}{b^2 + c} = 1$, where c is an arbitrary constant and this equation represents orthogonal trajectories of given conic.

Note

For these types of system as in examples 18 or 21, the given conic known as self orthogonal conic.

Exercise

1. **Determine order and degree of the following differential equation :**

 (i) $\left(\frac{d^2y}{dx^2}\right)^3 + x^2\left(\frac{dy}{dx}\right)^4 = 4$ **[CP 2002]**

 (ii) $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 1 + x$ **[CP 1999]**

 (iii) $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ **[CP 1998]**

 (iv) $\frac{d^2y}{dx^2} = x^3\left(\frac{dy}{dx}\right)^2$ **[CP 1986]**

 (v) $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{2}{3}} = \frac{d^2y}{dx^2}$ **[CP 1997]**

 (vi) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$

 (vii) $y = x\left(\frac{dy}{dx}\right) + \left(\frac{dx}{dy}\right)^2$

 (viii) $\sqrt{\frac{dy}{dx}}\sqrt{\frac{d^3y}{dx^3}} = 4$

2. **Formulate the differential equation from the given relations by eliminating the parameters :**

 (i) $V = \frac{A}{r} + B$, A and B are parameters. **[CP 1982, 1994; BP 1988]**

 (ii) $ax + by + c = 0$, where a and c are arbitrary constants and $b \neq 0$. **[BP 1993]**

 (iii) $y = A\sin x + B\cos x + x\sin x$ (A, B are parameters) **[CP 1993, 1996]**

 (iv) $y = e^{-x}(A\cos x + B\sin x)$ [A, B are parameters] **[CP 1996, 2003]**

 (v) $e^{y-x} = \lambda(y + x)$, λ being a parameter. **[CP 1981]**

 (vi) $y = A\cos(px - B)$, where p is constant and A, B parameters. **[CP 2000]**

3. Obtain the differential equation of the family of conics $ax^2 + by^2 = 1$, where a and b are arbitrary constants. **[BU 1998]**

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4. Obtain the differential equation of the family of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, where λ is the arbitrary parameter and a, b are two given constants.

5. Find the differential equation of the family of straight lines which are inclined at an angle α with the x-axis. [CP 1984, 88]

6. Find the family of curves cutting the family of parabolas $y^2 = 4ax$ at 45°.

7. Show that $xy = ae^x + be^{-x} + x^2$ is a solution of the differential equation

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$$

8. Show that $y = mx + \frac{a}{m}$ is a solution of the differential equation

$$y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}}$$

9. Show that $y = c_1e^{2x} + c_2e^{-x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

10. Find the degree of the differential equation satisfying the relation

$$\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right)$$

[*Hint* : put $x = \tan\theta$ and $y = \tan\phi$]

11. Find the differential equation of all non-horizontal lines in a plane.

12. If the differential equation $\frac{dy}{dx} = \frac{ax+b}{cx+d}$ represents a parabola, then find the values of a and c.

13. Show that $x = c_1\cos(nt + c_2) + \frac{c}{n^2-k^2}\sin kt$ is a solution of the differential equation $\frac{d^2x}{dt^2} + n^2x = c\sin kt$

14. Show that $y + x + 1 = 0$ is a solution of the differential equation $(y - x)dy - (y^2 - x^2)dx = 0$.

15. Show that

$$y = a\left(x+\sqrt{x^2-1}\right)^n + b\left(x-\sqrt{x^2-1}\right)^n$$ is

a solution of the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

16. Show that $y = e^{m\cos^{-1}x}$ satisfies the differential equation $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

17. Show that $2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$ satisfies the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$$

18. Find the orthogonal trajectories of each of the family of curves :

(i) $x^2 + y^2 = r^2$, where r being the variable parameter.

(ii) $y^2 = 4ax$, where a being the variable parameter. [CP 1999]

(iii) $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$, where λ being the variable parameter. [CP 1995, 2000, 2005]

(iv) $y = mx$, where m being the variable parameter.

(v) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, where a being the variable parameter.

(vi) $xy = c^2$, where c being the variable parameter.

19. Find the orthogonal trajectory of each of the following family of curve [c, a parameters]

(i) $r = c\sin^2\theta$ (ii) $r^2 = a^2\cos 2\theta$

(iii) $r = a(1 - \cos\theta)$ (iv) $r = a\theta$

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20. Show that the family of parabolas $y^2 = 4a(x + a)$ [where a being a parameter] is self orthogonal.

21. Show that the orthogonal trajectories of the family of confocal and coaxial parabolas $r = \dfrac{2a}{1+\cos\theta}$ is $r = \dfrac{2c}{1-\cos\theta}$

Answers

1. (i) order 2, degree 3
(ii) order 1, degree 2
(iii) order 1, degree 2
(iv) order 2, degree 1
(v) order 2, degree 3
(vi) order 2, degree not defined
(vii) order 1, degree 4
(viii) order 3, degree 1

2. (i) $\dfrac{d^2V}{dr^2} + \dfrac{2}{r}\dfrac{dV}{dr} = 0$

(ii) $\dfrac{d^2y}{dx^2} = 0$ (iii) $\dfrac{d^2y}{dx^2} + y = 2\cos x$

(iv) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$

(v) $\dfrac{dy}{dx} = \dfrac{1-y-x}{y+x-1}$

(vi) $\dfrac{d^2y}{dx^2} + p^2y = 0$

3. $xy\dfrac{d^2y}{dx^2} + \left(x\dfrac{dy}{dx} - y\right)\dfrac{dy}{dx} = 0$

4. $(a^2 - b^2)\dfrac{dy}{dx} = \left(x\dfrac{dy}{dx} - y\right)\left(x + y\dfrac{dy}{dx}\right)$

5. $\dfrac{dy}{dx} = \tan\alpha$

6. $\log|2x^2 - xy + y^2| = \dfrac{6}{\sqrt{7}}\tan^{-1}\dfrac{2y-x}{\sqrt{7}x}$ $+ c$, c being an arbitrary constant.

10. degree 1

11. $\dfrac{d^2x}{dy^2} = 0$

12. $a = 0, c \neq 0$

18. (i) $y = cx$ (ii) $2x^2 + y^2 = c^2$
(iii) $x^2 + y^2 - 2a^2\log x = c$
(iv) $x^2 + y^2 = c^2$
(v) $y^{\frac{4}{3}} - x^{\frac{4}{3}} = c$
(vi) $x^2 - y^2 = k^2$

19. (i) $r^2 = a\cos\theta$ (ii) $r^2 = k^2\sin^2\theta$
(iii) $r = c(1 + \cos\theta)$
(iv) $r = e^{\frac{1}{2}(c^2-\theta^2)}$

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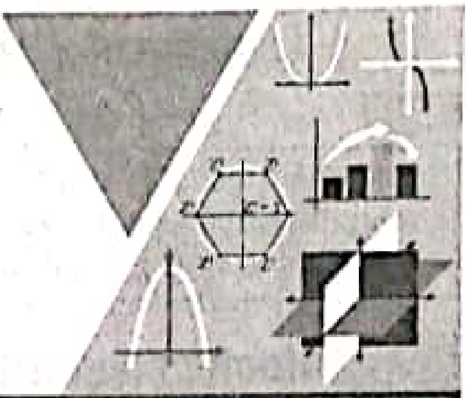
2 CHAPTER

Simple Methods to Solve First Order Differential Equation



Introduction

The general form of a first order first degree differential equation is $\frac{dy}{dx} = f(x, y)$ ... (1)

The form of the differential equation (1) is equivalent to $M dx + N dy = 0$ ... (2)

where M and N are either function of x and y or constants only.

Before progressing to determine the solution of the differential equation (1) (or (2)) first we state the existence and uniqueness criteria of the solution of the differential equation (1) (or (2)).

1. **Existence criteria :** The solution of the differential equation $\frac{dy}{dx} = f(x, y)$ exists if the function $f(x, y)$ is uniformly bounded, continuous and a single valued function over a domain D in the xy plane.
2. **Uniqueness criteria :** The solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is unique, if it satisfies the Picard's theorem which states that the solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is a unique integral curve through every point (x, y) in the interior of a closed rectangular domain D in xy plane, if the function $f(x, y)$ and $\frac{\partial f}{\partial y}$ both are continuous functions on the domain D.

Now we are going to elaborate some simple solution techniques of a first order first degree differential equation. The main types which are to be used for solution are (i) separation of variables, (ii) homogeneous equations, (iii) exact equation and (iv) linear equation. However, in this chapter solution procedure for first two types are discussed and rest two are kept for latter chapters.

2.1 Solution by Separation of Variable

If the differential equation $\frac{dy}{dx} = f(x, y)$ can be put into the form $f_1(x)dx + f_2(y)dy = 0$ [where $f_1(x)$ and $f_2(y)$ are functions of x and y respectively] then by direct integration we get the general solution which is $\int f_1(x)dx + \int f_2(y)dy = c$, where c is an arbitrary constant.

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Note

Sometimes a suitable transformation on dependent variable is required to reduce the given differential equation in separation of variable form and then by direct integration, one can get desired general solution. [see worked out example 16]

2.2 Homogeneous Equation

We know that a function $f(x, y)$ is said to be a homogeneous function of degree n if $f(x, y) = x^n\phi_1\left(\frac{y}{x}\right)$ or $f(x, y) = y^n\phi_2\left(\frac{x}{y}\right)$.

A differential equation of first order and first degree of the form $\frac{dy}{dx} = f(x, y)$, which can be reduced in the form $Mdx + Ndy = 0$ is said to be a homogeneous differential equation if M and N both are homogeneous functions of x and y of same order.

In this case, to solve let us substitute $y = vx$, where v is another variable and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Since M and N both are homogeneous functions of same degree then the given differential equation $\frac{dy}{dx} = f(x, y)$ can be written as $\frac{dy}{dx} = \psi\left(\frac{y}{x}\right)$ ... (3)

Using the substitutions, (3) can be reduced to $v + x\frac{dv}{dx} = \psi(v)$ ... (4)

Differential equation (4) can easily be solved by separation of variable technique, since from (4) we have $\frac{dv}{\psi(v) - v} = \frac{dx}{x}$ and on integration we get required general solution.

2.3 Equation Reducible to Homogeneous Form

We consider a first order first degree differential equation as follows :

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \quad \text{... (5)}$$

- **Case-1 :** Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{k}$, where k is a constant.

  Then substitute, $z = a_1x + b_1y$, so that $\frac{dz}{dx} = a_1 + b_1\frac{dy}{dx}$ and we have from (5)

  $\frac{1}{b_1}\left(\frac{dz}{dx} - a_1\right) = \frac{z + c_1}{kz + c_2}$ i.e. $\frac{dz}{dx} = \frac{b_1(z + c_1)}{kz + c_2} + a_1$ i.e. $dx = \frac{dz}{\frac{b_1(z+c_1)}{kz+c_2} + a_1}$, on integration whose general solution becomes $x + c = \int \frac{dz}{\frac{b_1(z+c_1)}{kz+c_2} + a_1}$, c being an arbitrary constant.

- **Case-2 :** Let $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

  In this case first find the point of intersection between the two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Let (α, β) be the poi[illegible]e those two straight lines meet. Then take the transformation as $u = x - \alpha$ and [illegible], so [illegible] $= dx$ and $dv = dy$.

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With these transformations differential equation (5) reduces to

$$\frac{dv}{du} = \frac{a_1(x-\alpha)+b_1(y-\beta)}{a_2(x-\alpha)+b_2(y-\beta)} = \frac{a_1u+b_1v}{a_2u+b_2v} \quad \dots (6)$$

Differential equation (6) is in homogeneous form and so to solve one can use the transformation $v = uz$, where z is a variable and then proceed same as earlier case to obtain the required general solution.

Notes

1. If the differential equation is same form of (5) but both numerator and denominator are of higher order, then also using suitable transformations, separation of variable technique can be applied [see illustrative example 10].
2. To solve a differential equation of the form $\frac{dy}{dx} = f(ax + by + c)$, one has to substitute, $z = ax + by + c$ and then proceed as earlier.

Illustrative Examples

1. Solve : $\frac{dy}{dx} + \frac{y(y-1)}{x(x-1)} = 0$

Solution $\frac{dy}{dx} + \frac{y(y-1)}{x(x-1)} = 0$ *i.e.,* $\frac{dx}{x(x-1)} + \frac{dy}{y(y-1)} = 0 \quad \dots (1)$

Now integrating (1), we have,

$$\int \frac{dx}{x(x-1)} + \int \frac{dy}{y(y-1)} = \log c, \text{ where } \log c \text{ is an arbitrary constant}$$

i.e., $\int \frac{dx}{x-1} - \int \frac{dx}{x} + \int \frac{dy}{y-1} - \int \frac{dy}{y} = \log c$

i.e., $\log(x-1) - \log x + \log(y-1) - \log y = \log c$ *i.e.,* $\log \frac{(x-1)(y-1)}{xy} = \log c$

Therefore $(x-1)(y-1) = cxy$, which is the required general solution.

2. Solve : $2xy\,dx - (x^2 - y^2)\,dy = 0$ [CP 1998]

Solution The given differential equation is $2xy\,dx - x^2dy + y^2dy = 0$

i.e., $\frac{2xy\,dx - x^2dy}{y^2} + dy = 0$ *i.e.,* $d\left(\frac{x^2}{y}\right) + d(y) = 0 \quad \dots (1)$

Now integrating (1), we have, $\frac{x^2}{y} + y = c$, where c is an arbitrary constant.

i.e., $x^2 + y^2 = cy$, which is the required general solution.

3. Solve : $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

Solution Here, the given differential equation is $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

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i.e., $\dfrac{x\,dx}{\sqrt{1-x^2}} + \dfrac{y\,dx}{\sqrt{1-y^2}} = 0$ ... (1) [Note : It can easily be solved by putting $y = vx$]

Now integrating (1), we have $\int \dfrac{d(1-x^2)}{2\sqrt{1-x^2}} + \int \dfrac{d(1-y^2)}{2\sqrt{1-y^2}} = c$, where c is an arbitrary constant

i.e., $\sqrt{1-x^2} + \sqrt{1-y^2} = c$

Which is the required general solution.

4. Solve : $\left(1 + 3e^{\frac{x}{y}}\right)dx + 3e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

Solution Here the given differential equation is $\left(1 + 3e^{\frac{x}{y}}\right)dx + 3e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

i.e., $\dfrac{dx}{dy} = \dfrac{3e^{\frac{x}{y}}\left(\frac{x}{y} - 1\right)}{1+3e^{\frac{x}{y}}}$ ... (1)

To solve (1), let us substitute $x = vy$, so that $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Now from (1), $v + y\dfrac{dv}{dy} = \dfrac{3e^v(v-1)}{1+3e^v}$ i.e., $y\dfrac{dv}{dy} = \dfrac{3e^v(v-1)}{1+3e^v} - v$

i.e., $y\dfrac{dv}{dy} = -\dfrac{3e^v + v}{1+3e^v}$ i.e., $\dfrac{(1+3e^v)dv}{v+3e^v} = -\dfrac{dy}{y}$ ... (2)

Now integrating (2), we have $\int \dfrac{(1+3e^v)dv}{v+3e^v} = -\int \dfrac{dy}{y} + \log c$, where c is an arbitrary constant

i.e., $\log(v + 3e^v) = -\log y + \log c$ i.e., $\log(v + 3e^v) = \log\dfrac{c}{y}$ i.e., $v + 3e^v = \dfrac{c}{y}$

i.e., $\dfrac{x}{y} + 3e^{\frac{x}{y}} = \dfrac{c}{y}$ i.e., $x + 3ye^{\frac{x}{y}} = c$

Which is the required general solution of the given differential equation.

5. Solve : $(2 - xy)y\,dx - (2 + xy)x\,dy = 0$

Solution Here the given differential equation is $(2 - xy)y\,dx - (2 + xy)x\,dy = 0$

i.e., $2(y\,dx - x\,dy) - xy(y\,dx + x\,dy) = 0$ i.e., $2\left(\dfrac{dx}{x} - \dfrac{dy}{y}\right) - d(xy) = 0$ ... (1)

Now integrating, we have $2\int\dfrac{dx}{x} - 2\int\dfrac{dy}{y} - \int d(xy) = c$, where c is an arbitrary constant.

i.e., $2\log x - 2\log y - xy = c$ i.e., $2\log\dfrac{x}{y} - xy = c$, which is the required general solution.

6. Solve : $\dfrac{dy}{dx} = \dfrac{x+y+1}{3x+3y+1}$ [CP 2009, 2013, 2015]

Solution Here the given differential equation is $\dfrac{dy}{dx} = \dfrac{x+y+1}{3x+3y+1}$

i.e., $\dfrac{dy}{dx} = \dfrac{x+y+1}{3(x+y)+1}$ ... (1)

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To solve (1), let us substitute $v = x + y$

Then $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ i.e., $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Thus (1) reduces to $\left(\frac{dv}{dx} - 1\right) = \frac{v+1}{3v+1}$

i.e., $\frac{dv}{dx} = \frac{v+1}{3v+1} + 1 = \frac{4v+2}{3v+1}$ i.e., $\frac{3v+1}{4v+2}dv = dx$

Integrating, we have $x + c = \int \frac{3v+1}{4v+2}dv$, c is an arbitrary constant

i.e., $x + c = \frac{3}{4}\int \frac{v+\frac{1}{3}}{v+\frac{1}{2}}dv = \frac{3}{4}\int \frac{v+\frac{1}{2}-\frac{1}{6}}{v+\frac{1}{2}}dv = \frac{3}{4}\left[\int dv - \frac{1}{6}\int \frac{dv}{v+\frac{1}{2}}\right] = \frac{3}{4}v - \frac{1}{8}\log\left(v + \frac{1}{2}\right)$

$= \frac{3}{4}(x + y) - \frac{1}{8}\log\left(x + y + \frac{1}{2}\right)$, since $x + y = v$

i.e., $\frac{x}{4} - \frac{3y}{4} + c = -\frac{1}{8}\log(2x + 2y + 1) + \frac{1}{8}\log 2$

i.e., $2(x - 3y) + \log(2x + 2y + 1) + c_1 = 0$, where $c_1 = c - \frac{1}{8}\log 2$, arbitrary constant

Which is the required general solution of the given diferential equation.

7. **Solve : $(3x + 2y - 5)dx + (2x + 3y - 5)dy = 0$** [CP 2015]

Solution Here the given differential equation is $(3x + 2y - 5)dx + (2x + 3y - 5)dy = 0$

i.e., $\frac{dy}{dx} + \frac{3x+2y-5}{2x+3y-5} = 0$ ... (1)

We see that the straight lines $3x + 2y - 5 = 0$ and $2x + 3y - 5 = 0$ intersects at the point (1, 1).

Therefore, to solve (1), we substitute $u = x - 1$ and $v = y - 1$ so that $\frac{dy}{dx} = \frac{dv}{du}$

Then we have from (1), $\frac{dv}{du} + \frac{3u+2v}{2u+3v} = 0$ ... (2)

To solve (2), we substitute $v = uz$, then $\frac{dv}{du} = z + u\frac{dz}{du}$

With this substitution (2) reduces to $z + u\frac{dz}{du} + \frac{3u+2uz}{2u+3uz} = 0$

i.e., $z + u\frac{dz}{du} + \frac{3+2z}{2+3z} = 0$ i.e., $\frac{3z^2+4z+3}{2+3z} + u\frac{dz}{du} = 0$ i.e., $\frac{du}{u} + \frac{3z+2}{3z^2+4z+3}dz = 0$

Now integrating, we have $\log u + \frac{1}{2}\log(3z^2 + 4z + 3) = \frac{1}{2}\log c$, where c is an arbitrary constant.

i.e., $\log[u^2(3z^2 + 4z + 3)] = \log c$

i.e., $u^2\left(3\frac{v^2}{u^2} + 4\frac{v}{u} + 3\right) = c$, since $v = uz$

i.e., $3u^2 + 4uv + 3v^2 = c$ i.e., $3(x-1)^2 + 4(x-1)(y-1) + 3(y-1)^2 = c$, since $x - 1 = u, y - 1 = v$

Which is the required general solution of the given differential equation.

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Alternative Method

Here given differential equation is $(3x + 2y - 5)dx + (2x + 3y - 5)dy = 0$

i.e., $3xdx + 3ydy + 2(ydx + xdy) - 5dx - 5dy = 0$

i.e., $3xdx + 3ydy + 2d(xy) - 5dx - 5dy = 0$

Now integrating, we have $3\int xdx + 3\int ydy + 2\int d(xy) - 5\int dx - 5\int dy = c$, where c is an arbitrary constant.

i.e., $3\frac{x^2}{2} + 3\frac{y^2}{2} + 2xy - 5x - 5y = c$, .

i.e., $3x^2 + 3y^2 + 4xy - 10x - 10y = c'$, where $c' = 2c$, where c' is an arbitrary constant

Which is the required general solution.

8. **Solve :** $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$ [CP 2017]

Solution Here the given differential equation is $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2-x^2}$ ... (1)

To solve (1), let us substitute $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then (1) reduces to $x^3\left(v + x\frac{dv}{dx}\right) = v^3x^3 + v^2x^2\sqrt{v^2x^2 - x^2}$

i.e., $v + x\frac{dv}{dx} = v^3 + v^2\sqrt{v^2-1}$ [since $x \neq 0$]

i.e., $x\frac{dv}{dx} = v^3 + v^2\sqrt{v^2-1} - v$ i.e., $\frac{dx}{x} = \frac{1}{v(v^2-1+v\sqrt{v^2-1})}dv$ ... (2)

Now, $\frac{1}{v(v^2-1+v\sqrt{v^2-1})} = \frac{1}{v\sqrt{v^2-1}(v+\sqrt{v^2-1})} = \frac{v-\sqrt{v^2-1}}{v\sqrt{v^2-1}(v+\sqrt{v^2-1})(v-\sqrt{v^2-1})}$

$= \frac{v-\sqrt{v^2-1}}{v\sqrt{v^2-1}(v^2-v^2+1)} = \frac{v-\sqrt{v^2-1}}{v\sqrt{v^2-1}} = \frac{1}{\sqrt{v^2-1}} - \frac{1}{v}$

So, form (2), we have $\frac{dx}{x} = \frac{dv}{\sqrt{v^2-1}} - \frac{dv}{v}$

Now integrating, we have $\log cx = \log(v + \sqrt{v^2-1}) - \log v$, where c is an arbitrary constant.

i.e., $cx = \frac{v+\sqrt{v^2-1}}{v} = \frac{\left(\frac{y}{x}\right)+\sqrt{\frac{y^2}{x^2}-1}}{\frac{y}{x}}$ [since $v = \frac{y}{x}$]

$= \frac{y+\sqrt{y^2-x^2}}{y}$

i.e., $y + \sqrt{y^2-x^2} = cxy$

Which is the required general solution.

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9. Solve : $(2x + y + 3)dx = (2y + x + 1)dy$ [CP 2014]

Solution Here, the given differential equation is $(2x + y + 3)dx = (2y + x + 1)dy$... (1)

To solve (1), we first find the point where the straight lines $2x + y + 3 = 0$ and $2y + x + 1 = 0$ meets. We see that the intersecting point is $\left(-\frac{5}{3}, \frac{1}{3}\right)$.

Then we substitute, $u = x + \frac{5}{3}$ and $v = y - \frac{1}{3}$, so that $dx = du$ and $dy = dv$

With these transformations (1) reduces to

$$(2u + v)du = (2v + u)dv \quad i.e., \frac{dv}{du} = \frac{2u+v}{2v+u} \quad ... (2)$$

Equation (2) is a homogeneous first order differential equation.

Therefore, to solve (2), we put $v = uz$, so that $\frac{dv}{du} = z + u\frac{dz}{du}$

Then (2) reduces to, $z + u\frac{dz}{du} = \frac{2u+uz}{2uz+u} = \frac{2+z}{2z+1}$, since $u \neq 0$

$$i.e., u\frac{dz}{du} = \frac{2+z}{2z+1} - z = \frac{2+z-2z^2-z}{2z+1} = \frac{2(1-z^2)}{2z+1} \quad i.e., \frac{2z+1}{2(1-z^2)}dz = \frac{du}{u}$$

$$i.e., \frac{du}{u} + \frac{1}{2}\frac{2z}{z^2-1}dz + \frac{1}{2}\frac{dz}{z^2-1} = 0$$

Integrating, we have $\log u + \frac{1}{2}\log(z^2-1) + \frac{1}{4}\log\frac{z-1}{z+1} = \log c$, where c is an arbitrary constant.

$$i.e., u\sqrt{z^2-1}\left(\frac{z-1}{z+1}\right)^{\frac{1}{4}} = c \quad i.e., u^4(z^2-1)^2\left(\frac{z-1}{z+1}\right) = c^4$$

$i.e., u^4(z^2-1)(z-1)^2 = c_1$, where $c_1 = c^4$, another arbitrary constant

$$i.e., u^4\left(\frac{v^2}{u^2}-1\right)\left(\frac{v}{u}-1\right)^2 = c_1 \quad i.e., (v^2-u^2)(v-u)^2 = c_1$$

$i.e., \left\{\left(y-\frac{1}{3}\right)^2 - \left(x+\frac{5}{3}\right)^2\right\}(y-x-2)^2 = c_1$, which is the required general solution.

10. Solve : $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ [CP 2014]

Solution Here the given equation is $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$$i.e., \frac{dy}{dx} + \frac{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = 0 \quad ... (1)$$

To solve (1), we substitute, $u = x + \frac{1}{2}$ and $v = y + \frac{1}{2}$, so that $du = dx$ and $dv = dy$

Then (1) becomes $\frac{dv}{du} + \frac{v^2+\left(\frac{\sqrt{3}}{2}\right)^2}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2} = 0$ $i.e., \frac{dv}{v^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{du}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2} = 0$

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Integrating, we have $\tan^{-1}\frac{2v}{\sqrt{3}} + \tan^{-1}\frac{2u}{\sqrt{3}} = c$, c being an arbitrary constant.

i.e., $\tan^{-1}\frac{2y+1}{\sqrt{3}} + \tan^{-1}\frac{2x+1}{\sqrt{3}} = c$, which is the required general solution.

11. Solve : $\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6}$ [CP 2008, 2010, 2015]

Solution Here, the given equation is $\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6}$

i.e., $2xdy + 3ydy - 6dy = 6xdx - 2ydx - 7dx$

i.e., $2(xdy + ydx) - 6xdx + 3ydy - 6dy - 7dx = 0$

i.e., $6xdx - 3ydy - 2d(xy) + 7dx + 6dy = 0$

Now integraing, we have

$6\int xdx - 3\int ydy - 2\int d(xy) + 7\int dx + 6\int dy = c_1$, where c_1 is an arbitrary constant.

i.e., $3x^2 - \frac{3}{2}y^2 - 2xy + 7x + 6y = c_1$

i.e., $6x^2 - 3y^2 - 4xy + 14x + 12y = c$, where $c = 2c_1$, where c is an arbitrary constant.

Which is the required general solution.

12. Solve : $xy(x^2 - 2)dy - dx = 0$ [CP 2011]

Solution Here the given differential equation is $xy(x^2 - 2)dy - dx = 0$ ... (1)

From (1), we have $ydy - \frac{dx}{x(x^2-2)} = 0$ i.e., $ydy - \frac{1}{2}\frac{x^2-(x^2-2)}{x(x^2-2)}dx = 0$

i.e., $ydy - \frac{1}{2}\frac{x}{x^2-2}dx + \frac{1}{2}\frac{dx}{x} = 0$

Integrating, we have $\frac{y^2}{2} - \frac{1}{4}\log(x^2-2) + \frac{1}{2}\log x = \frac{1}{4}\log c$, where c is an arbitrary constant.

i.e., $2y^2 - \log\frac{x^2-2}{x^2} = \log c$ i.e., $2y^2 = \log\frac{c(x^2-2)}{x^2}$ i.e., $c(x^2 - 2) = x^2e^{2y^2}$

Which is the required general solution, for c being an arbitrary constant.

13. Solve : $\frac{dy}{dx} = \frac{4x-5y+3}{5x-6y+2}$ [CP 2007]

Solution Here the given differential equation is $\frac{dy}{dx} = \frac{4x-5y+3}{5x-6y+2}$

i.e., $4xdx - 5ydx + 3dx = 5xdy - 6ydy + 2dy$

i.e., $4xdx - 5(ydx + xdy) + 6ydy + 3dx - 2dy = 0$

i.e., $4xdx - 5d(xy) + 6ydy + 3dx - 2dy = 0$

Now integrating, we have

$4\int xdx - 5\int d(xy) + 6\int ydy + 3\int dx - 2\int dy = c$, where c is an arbitrary constant

i.e., $2x^2 - 5xy + 3y^2 + 3x - 2y = c$

Which is the required general solution of the given differential equation.

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14. **Solve : $(x + y + 1)dx = (2x + 2y + 1)dy$** [CH 1984]

Solution Here the given differential equation is $(x + y + 1)dx = (2x + 2y + 1)dy$ ... (1)

To solve (1), we substitute $z = x + y$, then $\frac{dz}{dx} = 1 + \frac{dy}{dx}$

So, from (1), we have $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+1}$ i.e., $\left(\frac{dz}{dx} - 1\right) = \frac{z+1}{2z+1}$

i.e., $\frac{dz}{dx} = \frac{z+1}{2z+1} + 1 = \frac{3z+2}{2z+1}$ i.e., $dx = \frac{2z+1}{3z+2}dz$

i.e., $dx = \frac{2}{3}\frac{3z+2-\frac{1}{2}}{3z+2}dz = \frac{2}{3}\left(1 - \frac{1}{6}\frac{3}{3z+2}\right)dz$

Integrating we get, $x + c = \frac{2z}{3} - \frac{1}{9}\log(3z + 2)$, where c is an arbitrary constant.

i.e., $x + c = \frac{2}{3}(x + y) - \frac{1}{9}\log(3x + 3y + 2)$, since $z = x + y$

i.e., $9x + 9c - 6x - 6y + \log(3x + 3y + 2) = 0$

i.e., $3x - 6y + \log(3x + 3y + 2) = c_1$ ... (2)

where $c_1 = -9c$, another arbitrary constant.

Hence (2) is the required general solution.

15. **Solve : $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$, when $x = 0, y = 1$**

Solution Here the given differential equation is $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$

i.e., $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0$ ... (1)

Integrating, we have $\sin^{-1}x + \sin^{-1}y = \sin^{-1}c$, c being an arbitrary constant.

i.e., $\sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \sin^{-1}c$ i.e., $x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$ ... (2)

Equation (2) is the required general solution.

Using the initial condition for when $x = 0, y = 1$, we have $c = 1$

Hence the required particular solution is $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$

16. **Solve : $x\frac{dy}{dx} - y = x\sqrt{x^2 + y^2}$** [CH 2003]

Solution Here the given equation is $x\frac{dy}{dx} - y = x\sqrt{x^2 + y^2}$ ... (1)

To solve (1), we put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

With these transformation (1) becomes as $x\left(v + x\frac{dv}{dx}\right) - vx = x\sqrt{x^2 + v^2x^2}$

i.e., $x^2\frac{dv}{dx} = x^2\sqrt{1+v^2}$ i.e, $dx = \frac{dv}{\sqrt{1+v^2}}$

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Integrating, we have $x = \log(v + \sqrt{1+v^2}) + \log c$, c being an arbitrary constant.

$$\text{i.e., } e^x = \left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right)c, \text{ since } v = \frac{y}{x}$$

i.e., $c\left(y + \sqrt{x^2+y^2}\right) = xe^x$, which is the required general solution for c being an arbitrary constant.

Note

The given problem is not homogeneous, still we use $y = vx$ for solutions.

Alternative Method

$$x\frac{dy}{dx} - y = x\sqrt{x^2+y^2} \quad \text{i.e., } xdy - ydx = x\sqrt{x^2+y^2}\,dx$$

$$\text{i.e., } \frac{xdy - ydx}{x^2} = \sqrt{1+\frac{y^2}{x^2}}\,dx \quad \text{i.e., } \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\frac{y^2}{x^2}}} = dx$$

Now integrating, we have $\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int dx + \log c$, where c is an arbitrary constant.

$$\text{i.e., } \log\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) = x + \log c = \log ce^x \quad \text{i.e., } \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = ce^x$$

i.e., $y + \sqrt{x^2+y^2} = cxe^x$ i.e., $c_1\left(y + \sqrt{x^2+y^2}\right) = xe^x$, where $c_1 = \frac{1}{c}$ is an arbitrary constant.

17. **Solve : $\frac{dy}{dx} + 1 = e^{x-y}$**

Solution Here the given differential equation is $\frac{dy}{dx} + 1 = e^{x-y}$

$$\text{i.e., } dy + dx = e^{x-y}dx \quad \text{i.e., } d(x+y) = \frac{e^{2x}}{e^{x+y}}dx \quad \text{i.e., } e^{x+y}d(x+y) = e^{2x}dx$$

Now integrating, we have $\int e^{x+y}d(x+y) = \int e^{2x}dx + \frac{c}{2}$, where c is an arbitrary constant

$$\text{i.e., } e^{x+y} = \frac{e^{2x}}{2} + \frac{c}{2} \quad \text{i.e., } 2e^y = e^x + ce^{-x}, \text{ which is the general solution.}$$

18. **Show that the curve for which the normal at every point passes through a fixed point is a circle.** [CP 1993, BP 1998]

Solution We have the equation of the normal at any point (x, y) to the curve $f(x, y) = 0$ is $(Y - y)\frac{dy}{dx} + (X - x) = 0$

If it passes through a fixed point (α, β), then we get

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$$(\beta - y)\frac{dy}{dx} + (\alpha - x) = 0 \quad i.e., (x - \alpha)dx + (y - \beta)dy = 0$$

Now integrating, we have $\int(x - \alpha)dx + \int(y - \beta)dy = \frac{1}{2}a^2$, where a is an arbitrary constant.

i.e., $\int(x - \alpha)d(x - \alpha) + \int(y - \beta)d(y - \beta) = \frac{1}{2}a^2$

i.e., $\frac{(x-\alpha)^2}{2} + \frac{(x-\beta)^2}{2} = \frac{a^2}{2}$ *i.e.*, $(x - \alpha)^2 + (y - \beta)^2 = a^2$

It represents a circle **[Proved]**.

19. **A man invests ₹ 5000 at the rate 6% per year interest being compounded continuously, when will the sum double itself, (given log2 = 0.693).** [CH 1988]

Solution Let a man invests initially ₹ x at the interest rate 6%. Since interest is compounded continuously, then if t denotes the independent variable associated with time, the differential equation becomes $\frac{dx}{dt} = \frac{6}{100}x$ ... (1)

From (1), we have $\frac{3}{50}dt = \frac{dx}{x}$

Integrating, we have $\frac{3}{50}t = \log x + \log c$, c being an arbitrary constant.

i.e., $\frac{3}{50}t = \log cx$ *i.e.*, $cx = e^{\left(\frac{3}{50}\right)t}$ ... (2)

Since initially the man invests ₹ 5000, so the initial condition is at $t = 0$, $x = 5000$.

So from (2), $5000c = 1$ *i.e.*, $c = \frac{1}{5000}$

Therefore, the solution (2) reduces to $\frac{x}{5000} = e^{\left(\frac{3}{50}\right)t}$ ... (3)

When his investment is double, then $x = 10000$ at $t = t_1$ [say]

Then from (3), $\frac{10000}{5000} = e^{\left(\frac{3}{50}\right)t_1}$ *i.e.*, $\frac{3}{50}t_1 = \log 2$ *i.e.*, $t_1 = \frac{50}{3} \times 0.693 = 11.55$ (Approx.)

Hence the required duration for becoming double of his investment is 11.55 years.

20. **Show that the general solution of the equation $\frac{dy}{dx} + Py = Q$ can be written in the form $y = c(u - v) + v$, where c is an arbitrary constant and u and v are the two particular solutions.** [CH 1995, 2001, VH 2005]

Solution Here the given differential equation is $\frac{dy}{dx} + Py = Q$ ... (1)

Since u and v are two solutions of (1), therefore,

$\frac{du}{dx} + Pu = Q$ ... (2) and $\frac{dv}{dx} + Pv = Q$ ... (3)

Subtracting (3) from (2), we have $\frac{d}{dx}(u - v) + P(u - v) = 0$ *i.e.*, $\frac{d(u - v)}{u - v} = -Pdx$ ... (4)

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Again subtracting (3) from (1), we have $\frac{d}{dx}(y - v) + P(y - v) = 0$

$$\text{i.e., } \frac{d(y-v)}{y-v} = -P\,dx \qquad \ldots (5)$$

Comparing the equation (4) and (5), we have $\frac{d(y-v)}{y-v} = \frac{d(u-v)}{u-v}$

Integrating we have, $\log(y - v) = \log(u - v) + \log c$, where c is an arbitrary constant.

i.e., $y - v = c(u - v)$ *i.e.*, $y = c(u - v) + v$, which is the required form, for c being an arbitrary constant. **[Proved]**

21. Find the equation of the curve for which the Cartesian subnormal is constant.

Solution Since we have Cartesian subnormal at any point (x, y) to the curve $f(x, y) = 0$ is $y\frac{dy}{dx}$

According to the given condition, $y\cdot\frac{dy}{dx} = a$ (constant); giving $y\,dy = a\,dx$

Now integrating, we get $\frac{y^2}{2} = ax + \frac{c}{2}$, where c is an arbitrary constant.

i.e., $y^2 = 2ax + c$, which is the required equation of the curve.

22. Find the curve in which the portion of the tangent included between the coordinate axes is bisected at the point of contact.

Solution Since we have the equation of the tangent at any point (x, y) to the curve $f(x, y) = 0$ is $(Y - y) = \frac{dy}{dx}(X - x)$

i.e., $Y - y = y_1(X - x)$, where $y_1 = \frac{dy}{dx}$

i.e., $Xy_1 - Y = xy_1 - y$ *i.e.*, $\frac{X}{\frac{xy_1 - y}{y_1}} + \frac{Y}{y - xy_1} = 1$

Accordingly, $\frac{xy_1 - y}{2y_1} = x$ *i.e.*, $xy_1 - y = 2xy_1$

i.e., $xy_1 = -y$ *i.e.*, $x\frac{dy}{dx} = -y$ *i.e.*, $\frac{dx}{x} + \frac{dy}{y} = 0$

Now integrating $\log x + \log y = \log c$, where c is an arbitrary constant

$xy = c$ and it represents a rectangular hyperbola.

Exercise

1. Solve : $x\sqrt{y}\,dx + (1 + y)\sqrt{1+x}\,dy = 0$
2. Solve : $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$
3. Solve : $(x + y)^2\frac{dy}{dx} = a^2$ **[CP 1990]**
4. Solve : $y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right)$, given $y = 1$ when $x = 1$
5. Solve : $y\,dx - x\,dy = xy\,dx$ **[CP 1998]**

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6. Solve : $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$ [CP 1998]

7. Solve : $e^{x-y}\,dx + e^{y-x}dy = 0$

8. Solve : $(x + 2y - 3)dx = (2x + y - 3)dy$ [CH 1980, 1985]

9. Solve : $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$ [BH 1991]

10. Solve : $\frac{dy}{dx} = \frac{y - x + 1}{y + x - 5}$ [CP 2013]

11. Solve : $(x - 3y + 4)dy + (7y - 5x)dx = 0$

12. Solve : $\frac{dy}{dx} = \sin\frac{2y}{x} + \frac{y}{x}$, given $y = \frac{\pi}{4}$ at $x = 1$.

13. Solve : $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$ [CH 1987]

14. Solve : $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$ [CH 1982, VH 1997]

15. Solve: $xy\frac{dy}{dx} = y^2 + (x + y)^2 e^{-\frac{y}{x}}$ [NHB 1987]

16. Solve : $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}(1 - \frac{x}{y})dy = 0$

17. Solve : $\frac{dy}{dx} = \sqrt{y - x}$

18. Solve : $\log\frac{dy}{dx} = 2x + 3y$

19. Solve : $\tan^{-1}\left(\frac{dy}{dx}\right) = x + y$

20. The product of the slope and the ordinate at any point (x, y) of an integral curve passing through the point $(5, 3)$ is equal to the abscissa at that point. Find the equation of the integral curve.

21. Show that the curve for which the normal at every point passes through the point $(2, 3)$ is a circle $(x - 2)^2 + (y - 3)^2 = a^2$, where a is an arbitrary constant.

22. Show that the equation of the curve for which the Cartesian subtangent is constant is $y = ce^{\frac{x}{a}}$, where c is an arbitrary constant and a is the given constant.

Answers

1. $(x - 2)\sqrt{1 + x} + (y + 3)\sqrt{y} = c$
2. $y = c(a + x)(1 - ay)$
3. $x + y = \tan\left(\frac{c + y}{a}\right)$
4. $3x + (1 + 2x)(y - 2) = 0$
5. $x = cye^x$
6. $x = cye^{\frac{x-y}{xy}}$
7. $e^{2x} + e^{2y} = c$
8. $x + y - 2 = c(x - y)^3$
9. $(x - y - 1)^2(x + y - 1)^5 = c$
10. $2\tan^{-1}\frac{y-2}{x-3} + \log(x^2 + y^2 - 6x - 4y + 13) = c_1$
11. $(5x - 3y - 10)^4 = c(x - y - 1)^2$
12. $y = x\tan^{-1}x^2$
13. $cy = e^{\frac{y}{x}}$
14. $y^2 = x^2 + y^2(cx - 1)^2$
15. $e^{\frac{y}{x}} = \left(\frac{y}{x} + 1\right)\log(cx)$
16. $x + ye^{\frac{x}{y}} = c$
17. $2\sqrt{y - x} + 2\log(\sqrt{y - x} - 1) = x + c$
18. $2e^{-3y} + 3e^{2x} = c$
19. $\cos(x + y) + \sin(x + y) = ce^{x-y}$
20. $x^2 - y^2 = 16$

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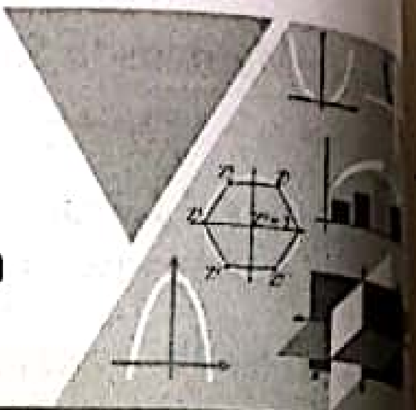
3 CHAPTER

First Order Exact Differential Equation



3.1 Exact Differential Equation

We consider a first order differential equation of the following form :

$$Mdx + Ndy = 0 \quad \text{... (1)}$$

where $M \equiv M(x, y)$ and $N \equiv N(x, y)$. Now if there exist a function $u \equiv u(x, y)$ such th $du = Mdx + Ndy$, then the differential equation (1) is said to be an exact differential equatio Thus if (1) is an exact differential equation, then from (1) we have $du = 0$ and on integratio we have, $u(x, y) = c$, which is the required general solution of the equation (1) [where c is a arbitrary constant].

■ **Condition for exactness :** The necessary and sufficient condition for a differenti equation of the form $Mdx + Ndy = 0$ to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

3.2 Solution Procedure of an Exact Differential Equation

We recall the differential equation (1) $M\,dx + Ndy = 0$

and consider that it is an exact so that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Now to solve (1) we discuss two processes as below :

- **Method-1 :** Let $u_1 = \int M dx$ (to evaluate assume y as constant) and $u_2 = \int N dy$ (to evaluate assume x as constant).

  Therefore the general solution is $u(x, y) = c$, where c is an arbitrary constant and $u(x, y)$ is the union of separate terms of the functions $u_1(x, y)$ and $u_2(x, y)$. This is illustrated in the next example.

- **Method-2 (By Inspection) :** Find a suitable function $u(x, y)$ such that $du = Mdx + Ndy$. Then from (1) we have, $du = 0$ and on integration, $u(x, y) = c$ [where c is an arbitrary constant], which is the required general solution.

Example : Solve $(y + e^x)dx + (x + e^y)dy = 0$

Here the given differential equation is $(y + e^x)dx + (x + e^y)dy = 0 \quad \text{... (2)}$

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Comparing (2) with its standard form $Mdx + Ndy = 0$, we have

$M = y + e^x$ and $N = x + e^y$. Now $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$ which ensures that the equation (2) is an exact differential equation.

- **Method-1** : We have, $u_1 = \int M dx = \int (y + e^x) dx = xy + e^x$ [Assuming y as constant]

and $u_2 = \int N dy = \int (x + e^y) dy = xy + e^y$ [Assuming x as constant]

Therefore $u(x, y)$ = union of separate terms of the functions u_1 and $u_2 = xy + e^x + e^y$

Thus the general solution of the given differential equation is $xy + e^x + e^y = c$, where c is an arbitrary constant.

- **Method-2** : We have $(y + e^x)dx + (x + e^y)dy = 0$

i.e., $ydx + xdy + e^x dx + e^y dy = 0$

i.e., $d(x{\cdot}y) + d(e^x) + d(e^y) = 0$ *i.e.*, $d(xy + e^x + e^y) = 0$

On integration we have, $xy + e^x + e^y = c$, which is the required general solution (where c is an arbitrary constant).

3.3 Equation can be Reducible to Exact Differential Equation

We consider the differential equation $ydx - xdy = 0$ ... (3)

By the property of exactness, it can be concluded that (3) is not an exact, since

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y) = 1$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x) = -1$

But by inspection we see that if we multiply (3) by $\frac{1}{x^2}$ then (3) will be an exact differential equation, since, $\frac{\partial}{\partial y}\left(\frac{y}{x^2}\right) = \frac{1}{x^2}$ and $\frac{\partial}{\partial x}\left(-\frac{x}{x^2}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$

Thus the differential equation (3) can be reduced to an exact differential equation and then it can be solved easily after multiplication of $\frac{1}{x^2}$ with the differential equation (3). The factor which is to be multiplied with a given differential equation to make it to exact differential equation is known as **Integrating Factor** (I.F.). For the differential equation (3), $\frac{1}{x^2}$ is an Integrating Factor (I.F.).

3.4 Rules to Determine Integrating Factor (I.F.)

We recall the differential equation (1), $Mdx + Ndy = 0$ and assume that it is not exact $\left(i.e., \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$, then the following rules (which one is appropriate) can be used to transform it to exact differential equation and then it can be solved.

- **Rule-1 :** Let M and N both are homogeneous functions of x and y [*i.e.*, $M = x^n\phi_1\left(\frac{y}{x}\right)$ and $N = x^n\phi_2\left(\frac{y}{x}\right)$] of same degree and $Mx + Ny \neq 0$. Then $\frac{1}{Mx + Ny}$ is an I.F. of the differential equation $Mdx + Ndy = 0$

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- **Rule-2 :** If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x only, say $\phi(x)$, then $e^{\int \phi(x)dx}$ is an I.F. of the differential equation $Mdx + Ndy = 0$
- **Rule-3 :** If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of y only, say $\psi(y)$, then $e^{-\int \psi(y)dy}$ is an I.F. of the differential equation $Mdx + Ndy = 0$
- **Rule-4 :** If the equation $Mdx + Ndy = 0$ can be re-written as $yf(xy)dx + xg(xy)dy = 0$ then $\frac{1}{Mx - Ny}$ (provided $Mx - Ny \neq 0$) is an I.F. of the equation $Mdx + Ndy = 0$
- **Rule-5 :** $x^h y^k$ is an I.F. of the equation $aydx + bxdy + x^m y^n(pydx + qxdy) = 0$, where h and k can be obtained from, $\frac{h+1}{a} = \frac{k+1}{b}$ and $\frac{h+m+1}{p} = \frac{k+n+1}{q}$.

Note

If one differential equation has an I.F., then it has infinite many I.F.

Illustrative Examples

1. Solve the equation $xdy - ydx = \cos\left(\frac{1}{x}\right)dx$ [CP 1997, 1999, 2009, BP 2000]

Solution Method-1 : Here the given differential equation is $xdy - ydx = \cos\left(\frac{1}{x}\right)dx$

i.e., $\left(y + \cos\frac{1}{x}\right)dx - xdy = 0$ ... (1)

Comparing (1) with its standard for $Mdx + Ndy = 0$, we have

$M = y + \cos\frac{1}{x}$ and $N = -x$; so that we have $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$.

Thus the given differential equation is not exact but can be reduced to an exact form.

Now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 - (-1) = 2$ and $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = -\frac{2}{x} = \phi(x)$ (say)

Thus the I.F. of (1) is $e^{\int \phi(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

Multiplying (1) by the I.F. $\frac{1}{x^2}$, we have $\frac{y+\cos\frac{1}{x}}{x^2}dx - \frac{1}{x}dy = 0$ ... (2)

Clearly (2) is an exact differential equation.

Now $\int \frac{\left(y+\cos\frac{1}{x}\right)}{x^2}dx = \int \frac{y}{x^2}dx + \int \frac{\cos\frac{1}{x}}{x^2}dx = y\int \frac{1}{x^2}dx - \int \cos\frac{1}{x}\, d\left(\frac{1}{x}\right)$ [Assuming y as constant]

$$= -\frac{y}{x} + \sin\left(\frac{1}{x}\right)$$

Also, $\int\left(-\frac{1}{x}\right)dy = -\frac{y}{x}$ [Assuming x as constant]

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Therefore the general solution of the given differential equation is $-\frac{y}{x} + \sin\left(\frac{1}{x}\right) = c$ where c is an arbitrary constant.

Method-2 (By inspection) : Here, the given differential equation is

$$x\,dy - y\,dx = \cos\left(\frac{1}{x}\right)dx \qquad \text{... (3)}$$

From (3), we have $\frac{x\,dy - y\,dx}{x^2} = \cos\left(\frac{1}{x}\right)\frac{dx}{x^2}$

$$i.e., -d\left(\frac{y}{x}\right) = -\cos\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right) \quad i.e., d\left(\frac{y}{x}\right) = \cos\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right)$$

On integrating, the general solution of the given differential equation is

$\frac{y}{x} = \sin\left(\frac{1}{x}\right) + c$, where c is an arbitrary constant.

2. **Show that the differential equation $(x^3 - 3x^2y + 2xy^2)dx - (x^3 - 2x^2y + y^2)dy = 0$ is exact. Hence solve it, given that $y = 1$ at $x = 1$.** [CP 2017]

Solution Here the given differential equation is

$$(x^3 - 3x^2y + 2xy^2)dx - (x^3 - 2x^2y + y^2)dy = 0 \qquad \text{... (1)}$$

Comparing (1) with its general form $M\,dx + N\,dy = 0$, we have

$$M = x^3 - 3x^2y + 2xy^2 \text{ and } N = -(x^3 - 2x^2y + y^2)$$

Now, $\frac{\partial M}{\partial y} = -3x^2 + 4xy$ and $\frac{\partial N}{\partial x} = -3x^2 + 4xy$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the given differential equation is exact.

Now, $\int M\,dx = \int(x^3 - 3x^2y + 2xy^2)dx = \frac{x^4}{4} - x^3y + x^2y^2$ [Assuming y as constant]

Also $\int N\,dy = \int -(x^3 - 2x^2y + y^2)dy = -x^3y + x^2y^2 + \frac{y^3}{3}$ [Assuming x as constant]

Therefore the general solution of the given differential equation is

$\frac{x^4}{4} - x^3y + x^2y^2 + \frac{y^3}{3} = c$, where c is an arbitrary constant.

Using the condition $y = 1$ at $x = 1$, we have $c = \frac{1}{4} - 1 + 1 + \frac{1}{3} = \frac{7}{12}$

Hence the general solution of the given differential equation for particular case is

$$\frac{x^4}{4} - x^3y + x^2y^2 + \frac{y^3}{3} = \frac{7}{12}$$

3. **Solve : $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$** [CP 2014]

Solution Here the given differential equation is

$$(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0 \qquad \text{... (1)}$$

Comparing (1) with the form $M\,dx + N\,dy = 0$, we have

$$M = \cos y + y\cos x \text{ and } N = \sin x - x\sin y$$

Now, $\frac{\partial M}{\partial y} = -\sin y + \cos x$ and $\frac{\partial N}{\partial x} = \cos x - \sin y$

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Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given differential equation is exact.

Now, we have $\int M dx = \int (\cos y + y\cos x) dx = x\cos y + y\sin x$ [Assuming y as constant]

Also $\int N dy = \int (\sin x - x\sin y) dy = y\sin x + x\cos y$ [Assuming x as constant]

Hence the required general solution of the given differential equation is $x\cos y + y\sin x =$
where c is an arbitrary constant.

4. **Solve : $y(1 + xy)dx + x(1 - xy)dy = 0$** [CP 2015, 200

Solution Here the given differential equation is $y(1 + xy)dx + x(1 - xy)dy = 0$ ... (1)

i.e., $yf(xy)dx + xf(xy)dy = 0$

Comparing (1) with the form $Mdx + Ndy = 0$, we have $M = y(1 + xy)$, $N = x(1 - xy)$

Now, $\frac{\partial M}{\partial y} = 1 + 2xy$ and $\frac{\partial N}{\partial x} = 1 - 2xy$. Thus (1) is not an exact equation.

We have, $M = yf(xy)$ and $N = xg(xy)$ where $f(xy) = 1 + xy$ and $g(xy) = 1 - xy$

Therefore, I.F. of (1) is $\frac{1}{Mx - Ny} = \frac{1}{xy(1 + xy) - xy(1 - xy)} = \frac{1}{2x^2y^2}$

Multiplying (1) by the I.F. $\frac{1}{2x^2y^2}$, we have $\frac{y(1 + xy)}{2x^2y^2} dx + \frac{x(1 - xy)}{2x^2y^2} dy = 0$

Now, $\int \frac{y(1 + xy)}{2x^2y^2} dx = \frac{1}{2y}\int \frac{dx}{x^2} + \frac{1}{2}\int \frac{dx}{x}$ [Assuming y as constant]

$$= -\frac{1}{2xy} + \frac{1}{2}\log x$$

Also $\int \frac{x(1 - xy)}{2x^2y^2} dy = \frac{1}{2x}\int \frac{dy}{y^2} - \frac{1}{2}\int \frac{dy}{y}$ [Assuming x as constant]

$$= -\frac{1}{2xy} - \frac{1}{2}\log y$$

Combining we have, the general solution of the given differential equation is

$-\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$, where c is an arbitrary constant.

i.e., $\frac{1}{xy} - \log\frac{x}{y} = c_1$ where $c_1 = -2c$.

5. **Solve : $\frac{\log(\sec x + \tan x)}{\cos x} dx = \frac{\log(\sec y + \tan y)}{\cos y} dy$** [CP 2015]

Solution Here, the given differential equations is

$$\frac{\log(\sec x + \tan x)}{\cos x} dx = \frac{\log(\sec y + \tan y)}{\cos y} dy$$

From (1), we have $\sec x \log(\sec x + \tan x)dx - \sec y \log(\sec y + \tan y)dy = 0$ ... (1)

i.e., $\left\{\frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} \log(\sec x + \tan x\right\} dx - \left\{\frac{\sec y(\sec y + \tan y)}{(\sec y + \tan y)} \log(\sec y + \tan y\right\} dy = 0$

i.e., $d(\log(\sec x + \tan x)) - d(\log(\sec y + \tan y)) = 0$

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i.e., $d(\log(\sec x + \tan x) - \log(\sec y + \tan y)) = 0$

Integrating and simplifying we have $\log \frac{\sec x + \tan x}{\sec y + \tan y} = c$, c being an arbitrary constant.

i.e., $\frac{\sec x + \tan x}{\sec y + \tan y} = c_1$, where $c_1 = e^c$, is another arbitrary constant,

which is the required general solution.

6. **Solve :** $xdx + ydy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$ [CP 2009, 2013]

Solution **Method-1 :** Here the given differential equation is $xdx + ydy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$

i.e., $\left(x - \frac{y}{x^2 + y^2}\right)dx + \left(y + \frac{x}{x^2 + y^2}\right)dy = 0$ ... (1)

Comparing (1) with its standard form $Mdx + Ndy = 0$

we have, $M = x - \frac{y}{x^2 + y^2}$, $N = y + \frac{x}{x^2 + y^2}$

Now, $\frac{\partial M}{\partial y} = -\frac{(x^2 + y^2)\cdot 1 - y\cdot 2y}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$ and $\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)\cdot 1 - x\cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore, the given differential equation is an exact differential equation.

Now, $\int M dx = \int\left(x - \frac{y}{x^2 + y^2}\right)dx = \frac{x^2}{2} - y\int\frac{dx}{x^2 + y^2}$ [Assuming y as constant]

$$= \frac{x^2}{2} - \tan^{-1}\frac{x}{y}$$

and $\int N dy = \int\left(y + \frac{x}{x^2 + y^2}\right)dy = \frac{y^2}{2} + x\int\frac{dy}{y^2 + x^2}$ [Assuming x as constant]

$$= \frac{y^2}{2} + \tan^{-1}\frac{y}{x} = \frac{y^2}{2} + \left(\frac{\pi}{2} - \tan^{-1}\frac{x}{y}\right) \quad \left[\text{since } \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{x}{y} = \frac{\pi}{2}\right]$$

$$= \frac{y^2}{2} - \tan^{-1}\frac{x}{y} + \frac{\pi}{2}$$

Combining we have, the required general solution of the given equation is

$\frac{x^2}{2} - \tan^{-1}\frac{x}{y} + \frac{y^2}{2} + \frac{\pi}{2} = c$, c being an arbitrary constant.

i.e., $x^2 + y^2 = c_1 + 2\tan^{-1}\frac{x}{y}$, where $c_1 = 2c - \pi$, another arbitrary constant.

Method-2 (By direct integration) : The given differential equation is

$$xdx + ydy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$$

i.e., $d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) - d\left(\tan^{-1}\frac{x}{y}\right) = 0$, since $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$

i.e., $d\left(x^2 + y^2 - 2\tan^{-1}\frac{x}{y}\right) = 0$

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Integrating, the required general solution of given differential equation is

$$x^2 + y^2 - 2\tan^{-1}\frac{x}{y} = c$$

i.e., $x^2 + y^2 = c + 2\tan^{-1}\frac{x}{y}$, where c is an arbitrary constant.

7. **Solve : $(x^2 - y^2)dx + 2xydy = 0$, given that $y = 4$, when $x = 1$.** [CP 20

Solution Here the given differential equation is $(x^2 - y^2)dx + 2xydy = 0$ ... (1)

Comparing (1) with its standard from $Mdx + Ndy = 0$, we have $M = x^2 - y^2$ and $N = 2xy$

Now $\frac{\partial M}{\partial y} = -2y$ and $\frac{\partial N}{\partial x} = 2y$. Clearly the given equation is not an exact one but can be reducible to an exact by multiplying a suitable I.F. We see that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-2y - 2y) = -\frac{2}{x} = \phi(x) \text{ (say)}$$

Therefore one I.F. of (1) is $e^{\int \phi(x)dx} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$

Now multiplying (1) by the I.F. $\frac{1}{x^2}$, we have, $\frac{x^2 - y^2}{x^2}dx + \frac{2y}{x}dy = 0$

Now, $\int \frac{x^2 - y^2}{x^2}dx = \int\left(1 - \frac{y^2}{x^2}\right)dx = \left(x + \frac{y^2}{x}\right)$ [Assuming y as constant]

and $\int \frac{2y}{x}dy = \frac{y^2}{x}$ [Assuming x as constant]

Combining, we have the required general solution of the given equation is

$$x + \frac{y^2}{x} = c \text{ i.e., } x^2 + y^2 = cx, \text{ where } c \text{ is an arbitrary constant}$$

Again using the condition $y = 4$, when $x = 1$, we have $1 + 16 = c$ i.e., $c = 17$

Therefore the required general solution for particular case is given by $x^2 + y^2 = 17x$.

8. **Solve : $xy(x^2 - 2)dy - dx = 0$**

Solution Here, the given differential equation is $xy(x^2 - 2)dy - dx = 0$ ... (1)

From (1), we have $ydy + \frac{dx}{x(2 - x^2)} = 0$

i.e., $ydy + \frac{1}{2}\frac{(2 - x^2) + x^2}{x(2 - x^2)}dx = 0$ i.e., $ydy + \frac{1}{2}\frac{dx}{x} + \frac{1}{2}\frac{x\,dx}{2 - x^2} = 0$

Integrating we have, $\frac{y^2}{2} + \frac{1}{2}\log x - \frac{1}{4}\log(x^2 - 2) = c$, c being arbitrary constant

i.e., $2y^2 + \log\frac{x^2}{x^2 - 2} = c_1$, where $c_1 = 4c$, another arbitrary constant

i.e., $x^2 = (x^2 - 2)e^{c_1 - 2y^2}$, which is the required general solution.

Note

Here I.F. $= \frac{1}{x(2 - x^2)}$

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9. Solve : $x\sin\frac{y}{x}dy = \left(y\sin\frac{y}{x} - x\right)dx$ [CP 2007]

Solution **Method-1 :** Here the given differential equation is $x\sin\frac{y}{x}dy = \left(y\sin\frac{y}{x} - x\right)dx$

i.e., $\left(y\sin\frac{y}{x} - x\right)dx - x\sin\frac{y}{x}dy = 0$ ... (1)

Comparing (1) with $Mdx + Ndy = 0$, we have $M = y\sin\frac{y}{x} - x$, $N = -x\sin\frac{y}{x}$

Now, $\frac{\partial M}{\partial y} = \sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x}$ and $\frac{\partial N}{\partial x} = -x\cos\frac{y}{x}\left(-\frac{y}{x^2}\right) - \sin\frac{y}{x} = -\sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x}$

Therefore the equation (1) is not exact.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x\sin\frac{y}{x}}\left(\sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x} + \sin\frac{y}{x} - \frac{y}{x}\cos\frac{y}{x}\right) = -\frac{2}{x} = \phi(x)$, say

Therefore, one I.F. of (1) is $e^{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$

Now multiplying (1) by the I.F. $= \frac{1}{x^2}$, we have $\frac{y\sin\frac{y}{x} - x}{x^2}dx - \frac{\sin\frac{y}{x}}{x}dy = 0$

Again, $\int\frac{y\sin\frac{y}{x} - x}{x^2}dx = -\int\frac{dx}{x} + y\int\frac{\sin\frac{y}{x}}{x^2}dx$ [Assuming y as constant]

$= -\log x - \int\sin\frac{y}{x}d\left(\frac{y}{x}\right) = -\log x + \cos\frac{y}{x}$

and $\int -\frac{\sin\frac{y}{x}}{x}dy = -\int\sin\frac{y}{x}d\left(\frac{y}{x}\right) = \cos\frac{y}{x}$ [Assuming x as constant]

Therefore the general solution of the given differential equation is $\cos\frac{y}{x} = c + \log x$, where c is an arbitrary constant.

Method-2 : $\left(y\sin\frac{y}{x} - x\right)dx - x\sin\frac{y}{x}dy = 0$

or, $\left(\frac{x\,dy - y\,dx}{x^2}\right)\sin\frac{y}{x} + \frac{dx}{x} = 0$ or, $\sin\frac{y}{x}d\left(\frac{y}{x}\right) + \frac{dx}{x} = 0$

Now integrating, $\int\sin\frac{y}{x}d\left(\frac{y}{x}\right) + \int\frac{dx}{x} = c$ or, $-\cos\frac{y}{x} + \log x = c$

or, $\cos\frac{y}{x} = \log x + c'$, where $c' = -c$ is an arbitrary constant.

10. Solve : $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$ [CP 2006]

Solution Here the given differential equation is

$(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$ ... (1)

Comparing (1) with $Mdx + Ndy = 0$, we have

$M = (x^2y^2 + xy + 1)y$ and $N = (x^2y^2 - xy + 1)x$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so given equation is not exact.

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Now $M = y(x^2y^2 + xy + 1) = yf(xy)$ and $N = x(x^2y^2 - xy + 1) = xg(xy)$

Therefore I.F. of (1) is $\dfrac{1}{Mx - Ny} = \dfrac{1}{xy(x^2y^2 + xy + 1) - xy(x^2y^2 - xy + 1)} = \dfrac{1}{2x^2y^2}$

Multiplying (1) by the I.F. $\dfrac{1}{2x^2y^2}$, we have $\dfrac{(x^2y^2 + xy + 1)y}{2x^2y^2}dx + \dfrac{(x^2y^2 - xy + 1)x}{2x^2y^2}dy = 0$

Now, $\displaystyle\int \frac{(x^2y^2 + xy + 1)y}{2x^2y^2}dx = \frac{y}{2}\int dx + \frac{1}{2}\int\frac{dx}{x} + \frac{1}{2y}\int\frac{dx}{x^2}$ [Assuming y as constant]

$$= \frac{xy}{2} + \frac{1}{2}\log x - \frac{1}{2xy}$$

Again, $\displaystyle\int \frac{(x^2y^2 - xy + 1)x}{2x^2y^2}dy = \frac{x}{2}\int dy - \int\frac{dy}{2y} + \frac{1}{2x}\int\frac{dy}{y^2}$ [Assuming x as constant]

$$= \frac{xy}{2} - \frac{1}{2xy} - \frac{1}{2}\log y$$

Combining, the general solution of the given differential equation is

$\dfrac{xy}{2} - \dfrac{1}{2xy} + \dfrac{1}{2}(\log x - \log y) = \dfrac{c}{2}$, where c is an arbitrary constant.

i.e., $xy - \dfrac{1}{xy} + \log\dfrac{x}{y} = c$.

11. **Examine whether the differential equation $(2x^3 + 4y)dx + (4x + y - 1)dy = 0$ is exact or not.** [CP 2013]

Solution Here the given differential equation is $(2x^3 + 4y)dx + (4x + y - 1)dy = 0$ ... (1)

Comparing (1) with the equation $Mdx + Ndy = 0$, we have

$M = 2x^3 + 4y$ and $N = 4x + y - 1$. Now $\dfrac{\partial M}{\partial y} = 4$ and $\dfrac{\partial N}{\partial x} = 4$.

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, therefore, the given differential equation is an exact differential equation.

12. **Show that $[x(x^2 - y^2)]^{-1}$ is an integrating factor of the differential equation $(x^2 + y^2)dx - 2xydx = 0$.**

Solution Here the given differential equation is $(x^2 + y^2)dx - 2xydx = 0$ ... (1)

Multiplying (1) by $[x(x^2 - y^2)]^{-1}$ we have, $\dfrac{x^2 + y^2}{x(x^2 - y^2)}dx - \dfrac{2xy}{x(x^2 - y^2)}dy = 0$ ... (2)

Comparing (2) with the equation $Mdx + Ndy = 0$,

we have $M = \dfrac{x^2 + y^2}{x(x^2 - y^2)}$ and $N = -\dfrac{2xy}{x(x^2 - y^2)}$

Now, $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{x^2 + y^2}{x(x^2 - y^2)}\right) = \dfrac{x(x^2 - y^2)\cdot 2y - (x^2 + y^2)(-2xy)}{x^2(x^2 - y^2)^2} = \dfrac{4xy}{(x^2 - y^2)^2}$

Also, $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(-\dfrac{2xy}{x(x^2 - y^2)}\right) = -\dfrac{x(x^2 - y^2)\cdot 2y - 2xy(3x^2 - y^2)}{x^2(x^2 - y^2)^2} = \dfrac{4xy}{(x^2 - y^2)^2}$

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Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the differential equation (2) is exact.

Hence $[x(x^2 - y^2)]^{-1}$ is an integrating factor of the given differential equation.

13. **Solve : $(x^2 + y^2 + 4)x\,dx + (x^2 - y^2 + 9)y\,dy = 0$** **[CP 2016]**

Solution Here the given differential equation is

$$(x^2 + y^2 + 4)x\,dx + (x^2 - y^2 + 9)y\,dy = 0 \quad \text{... (1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$ we have $M = (x^2 + y^2 + 4)x$, $N = (x^2 - y^2 + 9)y$

Now $\frac{\partial M}{\partial y} = 2xy$ and $\frac{\partial N}{\partial x} = 2xy$. Thus the given equation is exact.

Therefore, we have, $\int M\,dx = \int (x^2 + y^2 + 4)x\,dx = \frac{x^3}{3} + \frac{x^2y^2}{2} + 2x^2$ [Assuming y as constant]

and $\int N\,dy = \int (x^2 - y^2 + 9)y\,dy = \frac{x^2y^2}{2} - \frac{y^3}{3} + \frac{9}{2}y^2$ [Assuming x as constant]

Combining we have, the required general solution is

$$\frac{x^3}{3} + \frac{x^2y^2}{2} + 2x^2 - \frac{y^3}{3} + \frac{9}{2}y^2 = c,$$ where c is an arbitrary constant.

14. **Solve : $[xy\cos(xy) + \sin(xy)]dx + [x^2\cos(xy)]dy = 0$** **[CP 1988, 1996]**

Solution **Method-1 :** Here the given differential equation is

$$[xy\cos(xy) + \sin(xy)]dx + [x^2\cos(xy)]dy = 0 \quad \text{... (1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$, we have $M = xy\cos(xy) + \sin xy$ and $N = x^2\cos(xy)$

Now, $\frac{\partial M}{\partial y} = x\cos xy - x^2y\sin(xy) + x\cos xy = 2x\cos xy - x^2y\sin(xy)$

and $\frac{\partial N}{\partial x} = -x^2\sin(xy)(y) + 2x\cos xy = 2x\cos xy - x^2y\sin(xy)$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the given differential equation is exact.

Now, $\int M\,dx = \int (xy\cos(xy) + \sin xy)dx = \int xy\cos(xy)dx + \int \sin xy\,dx$ [Assuming y as constant]

$$= \frac{1}{y}\int (xy)\cos(xy)d(xy) + \frac{1}{y}\int \sin xy\,d(xy)$$

$$= \frac{1}{y}(xy\sin(xy)) - \frac{1}{y}\int \sin xy\,d(xy) + \frac{1}{y}\int \sin xy\,d(xy) = x\sin(xy)$$

and $\int N\,dy = \int x^2\cos(xy)dy = x\int \cos(xy)d(xy)$ [Assuming x as constant]

$$= x\sin(xy)$$

Hnece the general solution of the given differential equation is

$x\sin(xy) = c$, where c is an arbitrary constant.

Method-2 : Here, the given differential equation is $xy\cos(xy) + \sin(xy)dx + x^2\cos(xy)dy = 0$

i.e., $x\cos xy[y\,dx + x\,dy] + \sin xy\,dx = 0$

i.e., $x\cos(xy)d(xy) + \sin xy\,dx = 0$

i.e., $d(x\sin xy) = 0$

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Integrating we get, the general solution of the given differential equation is $x\sin(xy) = c$, where c is an arbitrary constant.

15. Solve : $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$ [CH 20

Solution Here the given differential equation is

$$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0 \quad \dots (1)$$

Comparing (1) with $Mdx + Ndy = 0$, we have,

$$M = 2xy^4e^y + 2xy^3 + y \text{ and } N = x^2y^4e^y - x^2y^2 - 3x$$

Therefore, $\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$ and $\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$

Thus the equation (1) is not an exact differential equation.

Now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3e^y + 8xy^2 + 4$ and

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{8xy^3e^y + 8xy^2 + 4}{y(2xy^3e^y + 2xy^2 + 1)} = \frac{4}{y} = \psi(y) \text{ (say)}$$

Therefore one I.F. is $e^{-\int \psi(y)dy} = e^{-\int \frac{4}{y}dy} = e^{-4\log y} = e^{\log y^{-4}} = \frac{1}{y^4}$

Multiplying by the I.F., the equation (1) reduces to

$$\left(2xe^y + 2\frac{x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - 3\frac{x}{y^4}\right)dy = 0$$

Now, $\int\left(2xe^y + 2\frac{x}{y} + \frac{1}{y^3}\right)dx = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}$ [Assuming y to be constant]

and $\int\left(x^2e^y - \frac{x^2}{y^2} - 3\frac{x}{y^4}\right)dy = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}$ [Assuming x to be constant]

Combining, the general solution is given by : $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$, where c is an arbitr constant.

16. Solve : $3ydx - 2xdy + x^2y^{-1}(10ydx - 6xdy) = 0$ [CP 19

Solution Here the given differential equation is

$$3ydx - 2xdy + x^2y^{-1}(10ydx - 6xdy) = 0 \quad \dots (1)$$

Let x^hy^k be an I.F. and the multiplying (1) by this I.F., we have

$$x^hy^k(3ydx - 2xdy) + x^hy^kx^2y^{-1}(10ydx - 6xdy) = 0 \quad \dots (2)$$

Therefore (2) is an exact differential equation and so we have

$$\frac{\partial}{\partial y}(3y^{k+1}x^h + 10x^{h+2}y^k) = \frac{\partial}{\partial x}(-2x^{h+1}y^k - 6x^{h+3}y^{k-1}) \quad \dots (3)$$

i.e., $3(k + 1)x^hy^k + 10kx^{h+2}y^{k-1} = -2(h + 1)x^hy^k - 6(h + 3)x^{h+2}y^{k-1}$

Comparing the corresponding coefficient of x^hy^k of (3) we have

$$3(k + 1) = -2(h + 1), \text{ i.e., } 2h + 3k = -5 \quad \dots (4)$$

and comparing coefficient of $x^{h+2}y^{k+1}$, we have

$$10k = -6(h + 3) \text{ i.e., } 6h + 10k = -18 \quad \dots (5)$$

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Solving (4) and (5), $h = 2$, $k = -3$.

Therefore x^2y^{-3} is an I.F. of the given equation (1).

Multiplying by x^2y^{-3}, the first expression reduces to

$$x^2y^{-3}(3y\,dx - 2x\,dy) = 3x^2y^{-2}dx - 2x^3y^{-3}dy = d(x^3y^{-2})$$

Again multiplying by x^2y^{-3}, the second expression reduces to

$$x^2y^{-3}\{x^2y^{-1}(10y\,dx - 6x\,dy)\} = 10x^4y^{-3}dx - 6x^5y^{-4}dy$$
$$= 2\{5x^4y^{-3}dx - 3y^{-4}x^5dy\} = 2d(x^5y^{-3})$$

Combining we have the required general solution is $x^3y^{-2} + 2x^5y^{-3} = c$, where c is an arbitrary constant.

17. **Solve : $(x + y)(dx - dy) = dx + dy$** [CP 1995]

Solution Here the given differential equation is $(x + y)(dx - dy) = dx + dy$

i.e., $dx - dy = \dfrac{d(x+y)}{x+y}$

Now integrating, we get $\int dx - \int dy = \int \dfrac{d(x+y)}{x+y}$ i.e., $x - y = \log(x + y) + c$

where c is an arbitrary constant.

18. **Solve : $(x^2y^2 + xy)y\,dx + (x^2y^2 - 1)x\,dy = 0$**

Solution Here, the given differential equation is $(x^2y^2 + xy)y\,dx + (x^2y^2 - 1)x\,dy = 0$

i.e., $xy(xy + 1)y\,dx + (xy - 1)(xy + 1)x\,dy = 0$

i.e., $xy^2dx + (xy + 1)x\,dy = 0$ i.e., $xy(y\,dx + x\,dy) - x\,dy = 0$

i.e., $d(xy) - \dfrac{dy}{y} = 0$

Now, integrating, we have $\int d(xy) - \int \dfrac{dy}{y} = c$ i.e., $xy - \log y = c$

Which is the required general solution for c being an arbitrary constant.

19. **Solve : $x\cos\left(\dfrac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\dfrac{y}{x}\right)(x\,dy - y\,dx)$**

Solution Here the given differential equation is $x\cos\left(\dfrac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\dfrac{y}{x}\right)(x\,dy - y\,dx)$

i.e., $\dfrac{x}{x^2}\cos\left(\dfrac{y}{x}\right)d(xy) = y\sin\left(\dfrac{y}{x}\right)\left(\dfrac{x\,dy - y\,dx}{x^2}\right)$

i.e., $\cos\left(\dfrac{y}{x}\right)\dfrac{d(xy)}{xy} = \sin\left(\dfrac{y}{x}\right)d\left(\dfrac{y}{x}\right)$ i.e., $\dfrac{d(xy)}{xy} = \tan\left(\dfrac{y}{x}\right)d\left(\dfrac{y}{x}\right)$

Now, integrating, we get $\int \dfrac{d(xy)}{xy} = \int \tan\left(\dfrac{y}{x}\right)d\left(\dfrac{y}{x}\right) + \log c$, c being an arbitrary constant.

i.e., $\log(xy) = \log\sec\left(\dfrac{y}{x}\right) + \log c$ i.e., $xy = c\sec\dfrac{y}{x}$

$\therefore\ xy\cos\dfrac{y}{x} = c$, which is the required general solution of the given differential equation, for c being an arbitrary constant.

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Exercise

1. State the necessary and sufficient condition for which the differential equation $f(x, y)\frac{dy}{dx} + g(x, y) = 0$ may be exact.

2. **State whether each of the following is true or false :**

   (i) The equation $xdy - ydx = 0$ is exact.

   (ii) The equation $e^x \sin y dx + (e^x + 1) \cos y dy = 0$

   (iii) The equaion $\left(xy^2 - e^{x^{-\frac{1}{3}}}\right)dx - x^2ydy = 0$ is exact.

   (iv) The equation $(x^2 + y^2 + 4)xdx + (x^2 - y^2 + 9)ydy = 0$ is exact.

3. **Find I.F. of the following differential equations :**

   (i) $xdy - ydx = 0$

   (ii) $(x^4 + y^4)dx - xy^3dy = 0$

   (iii) $y(1 + xy)dx + x(1 - xy)dy = 0$

   (iv) $y(2xy + 1)dx + x(1 + 2xy + x^2y^2)dy = 0$

   (v) $(2xy + e^x)ydx - e^x dy = 0$

   (vi) $y(1 + xy)dx - xdy = 0$ **[CP 1990]**

4. Show that x^2y is an I.F. of the differential equation $3ydx + 2xdy = 0$ **[CP 1991]**

5. Verify that x^3 is an I.F. of $2\sin y^2 dx + xy\cos y^2 dy = 0$

6. Express the differential equation $(x + y)dx + xdy = 0$ in the form $dz = 0$ and hence ditermine z in terms of x and y. **[CP 1987]**

7. Express the differential equation $(1 - x^2)dy - 2xydx = 0$ in the form $dz = 0$ and hence determine z in terms of x and y.

8. Solve : $(2x + 3y + 7)dx + (3x - 5y + 2)dy = 0$ **[CP 1990]**

9. Solve : $x\,dy - ydx + a(x^2 + y^2)dx = 0$ **[CP 1993]**

10. Solve : $(x + y)(dx - dy) = dx + dy$ **[CP 1995]**

11. Solve : $(1 + xy)ydx + (1 - xy)xdy = 0$ **[CP 1996]**

12. Solve : $ydx - xdy = xydx$ **[CP 1999]**

13. Solve : $(x^2 + y^2 + 4)xdx + (x^2 - y^2 + 9)ydy$ **[CP 1994]**

14. Solve : $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$ **[CP 1993, 2000]**

15. Solve : $y(2xy + 1)dx + x(1 + 2xy + x^2y^2)dx = 0$ **[CP 1998]**

16. Show that the differential equation $(y^3 + 3x^2y)dx + (x^3 + 3y^2x)dy = 0$ is exact. Hence solve it, given that $y = 1$ at $x = 1$. **[Similar CP 1992, 2005]**

17. Solve : $(xy^2 + 3e^{x^{-3}})dx - x^2ydy = 0$

18. Solve : $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

19. Solve : $x\frac{dy}{dx} + \cos x - y = 0$

20. If $x^{\alpha}y^{\beta}$ be an I.F. of the equation $(xydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$, then find the value of α and β.

Answers

1. $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$

2. (i) False (ii) True (iii) False (iv) True

3. (i) $\frac{1}{x^2}$ (ii) $\frac{1}{x^5}$ (iii) $\frac{1}{2x^2y^2}$ (iv) $-\frac{1}{x^3}$ (v) $\frac{1}{y^2}$ (vi) $\frac{1}{y^2}$

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6. $z = x^2 + 2xy$

7. $z = y(1 - x^2)$

8. $2x^2 + 6xy - 5y^2 + 14x + 4y = c$

9. $\tan^{-1}\left(\frac{y}{x}\right) - ax = c$

10. $x - y + c = \log(x + y)$

11. $x = cye^{\frac{1}{xy}}$

12. $x = cye^x$

13. $\frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{y^4}{4} + 2x^2 + \frac{9y^2}{2} = c$

14. $x\cos y + y\sin x = c$

15. $4xy + 1 = 2x^2y^2\log(cy)$

16. $xy(x^2 + y^2) = c,\ xy(x^2 + y^2) = 2$

17. $\frac{y^2}{x^2} = 2e^{\frac{1}{x^3}} + c$

18. $y(1 + x^2) = \frac{4}{3}x^3 + c$

19. $y + x\sin\frac{1}{x} = cx$

20. $\alpha = 1,\ \beta = 2$

[c being an arbitrary constant]

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4 CHAPTER

First Order Linear Differential Equation

Introduction

As we discussed in first chapter, if an ordinary differential equation of first order with one independent variable x and one dependent variable $y(x)$ can be written in the following form

$$\frac{dy}{dx} + Py = Q \qquad \ldots (1)$$

where P and Q are either constants or function of the independent variable x, is said to be a first order linear differential equation. In this chapter we aim to discuss the solution procedure of first order linear differential equation.

4.1 Solution Procedures

We are to solve the linear differential equation (1) where P and Q are either constants or function of x by making the left hand side of the differential equation (1) in exact form or to make it a total differential.

To make L.H.S. a total differential, we multiply it by an integrating factor (I.F.).

Here I.F. = $e^{\int P dx}$, then L.H.S. becomes $e^{\int P dx}\frac{dy}{dx} + Py \cdot e^{\int P dx} = \frac{d}{dx}\left(ye^{\int P dx}\right)$

Then (1) reduces to $\frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$ i.e., $d\left(ye^{\int P dx}\right) = \left(Qe^{\int P dx}\right)dx$

Integrating both sides, we get the required general solution which is as follows

$ye^{\int P dx} = \int Qe^{\int P dx}\,dx + c$, c being an arbitrary constant.

Note

> If P and Q both are constants then the linear differential equation can be written as $\frac{dy}{dx} = Q - Py$ and solving $\frac{dy}{Q - Py} = dx$, we get required solution.

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4.2 Equation Reducible to Linear Equation : Bernoulli's Equation

We consider a first order ordinary differential equation of the following form :

$$\frac{dy}{dx} + Py = Qy^n \quad \text{... (2)}$$

where P and Q are either constants or functions of x.

The ordinary differential equation (2) is known as Bernoulli's equation. Although it is not a linear differential equation, still it can be reduced it in linear form.

Now we have from (2), $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ...(3)

Let $z = y^{1-n}$, then $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ giving, $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$.

With these transformations, (3) becomes $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

$$i.e., \frac{dz}{dx} + (1-n)Pz = (1-n)Q \quad i.e., \frac{dz}{dx} + P_1z = Q_1 \quad \text{...(4)}$$

where $P_1 = (1-n)P$ and $Q = (1-n)Q$ are either constants or function of x.

Hence equation (4) is a linear differential equation of first order and can be solved easily by multiplying the I.F. $e^{\int P_1 dx}$ and then integrating.

Note

> The differential equation (1) is known as linear in y. However the form $\frac{dx}{dy} + P_2x = Q_2$, where P_2 and Q_2 are either constants or function of y only is a differential equation with linear on x. It can be solved by finding I.F. $= e^{\int P_2 dy}$.

Illustrative Examples

1. Solve : $\frac{dy}{dx} + \left(\frac{1}{1+x^2}\right)y = \frac{e^{\tan^{-1}x}}{1+x^2}$ [CP 1988]

Solution Here the given differential equation is

Here $\frac{dy}{dx} + \left(\frac{1}{1+x^2}\right)y = \frac{e^{\tan^{-1}x}}{1+x^2}$ ...(1)

Clearly the differential equation (1) is a first order linear differential equation. Therefore, its I.F. is $e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

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Multiplying both sides of (1) by the I.F. $e^{\tan^{-1}x}$ and then integrating, we have

$\int d(ye^{\tan^{-1}x}) = \int \frac{(e^{\tan^{-1}x})^2}{1+x^2} dx + c$, c being an arbitrary constant

i.e., $ye^{\tan^{-1}x} = \int e^{2z} dz + c$ [to evaluate R.H.S. we substitute $z = \tan^{-1}x$]

$$= \frac{e^{2z}}{2} + c = \frac{1}{2} e^{2\tan^{-1}x} + c, \text{ since } z = \tan^{-1}x$$

i.e., $y = \frac{1}{2}e^{\tan^{-1}x} + ce^{-\tan^{-1}x}$, where c is an arbitrary constant.

Which is the required general solution.

2. **Solve : $\cos^2 x \frac{dy}{dx} + y = \tan x$.** [CP 2001, 2008, 2011, 201

Solution Here the given differential equation is $\cos^2 x \frac{dy}{dx} + y = \tan x$

i.e., $\frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$ ...(1)

Clearly (1) is a first order linear differential equation.

So the I.F. of (1) is $e^{\int \sec^2 x dx} = e^{\tan x}$

Multiplying (1) by the I.F. $= e^{\tan x}$ and then integrating, we have

$ye^{\tan x} = \int e^{\tan x} \tan x \sec^2 x \, dx + c$, c being an arbitrary constant

i.e., $ye^{\tan x} = \int ze^z dz + c$ [To evaluate the integration in R.H.S. we substitute $z = \tan x$]

$$= ze^z - e^z + c$$

$$= \tan x e^{\tan x} - e^{\tan x} + c, \text{ since } z = \tan x$$

i.e., $y = \tan x - 1 + ce^{-\tan x}$, which is the required general solution for c being an arbitrary constant.

3. **Solve : $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$**

Solution Here the given differential equation is

$y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$ *i.e.*, $y^3 \frac{dx}{dy} + xy - 1 = 0$ *i.e.*, $\frac{dy}{dx} + \frac{1}{y^2}x = \frac{1}{y^3}$ ...(1)

Clearly the differential equation (1) is a first order differential equation with linear in x

Therefore the I.F. $= e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$

Multiplying (1) by the I.F. $e^{-\frac{1}{y}}$ and integrating we have,

$xe^{-\frac{1}{y}} = \int e^{-\frac{1}{y}} \cdot \frac{1}{y^3} dy + c$,

where c being an arbitrary constant. ...(2)

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To evaluate R.H.S. we substitute $z = -\frac{1}{y}$, so that $dz = \frac{dy}{y^2}$ and then we have from (2),

$$\int e^{-\frac{1}{y}} \frac{1}{y^3} dy = -\int ze^z dz = e^z - ze^z = e^{-\frac{1}{y}}\left(1+\frac{1}{y}\right).$$

Thus from (2), the general solution of the given equation reduces to

$xe^{-\frac{1}{y}} = e^{-\frac{1}{y}}\left(1+\frac{1}{y}\right) + c$ i.e., $x = \left(1+\frac{1}{y}\right) + ce^{\frac{1}{y}}$, where c is an arbitrary constant.

4. **Solve : $\sin x \frac{dy}{dx} - y\cos x + y^2 = 0$** **[CP 2011, 2014, 2016]**

Solution Here the given differential equation is $\sin x \frac{dy}{dx} - y\cos x + y^2 = 0$

i.e., $\frac{dy}{dx} - y\cot x = -y^2 \operatorname{cosec} x$ i.e., $y^{-2}\frac{dy}{dx} - y^{-1}\cot x = -\operatorname{cosec} x$ ... (1)

The differential equation (1) is in Bernoulli's form.

To reduce it in linear differential equation, we substitute $z = y^{-1}$, then $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$.

With these substitution (1) reduces to

$-\frac{dz}{dx} - z\cot x = -\operatorname{cosec} x$ i.e., $\frac{dz}{dx} + z\cot x = \operatorname{cosec} x$ ... (2)

Equation (2) in a first order linear differential equation.

Therefore its I.F. is $e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

Multiplying (2) with the I.F. $\sin x$ and then integrating, we have

$d(z\sin x) = \operatorname{cosec} x \sin x\, dx$

i.e., $z\sin x = \int dx + c$, c being an arbitrary constant

i.e., $\frac{\sin x}{y} = x + c$ i.e., $\sin x = y(x + c)$

Which is the required general solution.

5. **Solve : $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$** **[CP 1989, 1995, 2011, 2015; BP 2002, 2003]**

Solution Here the given differential equation is $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

i.e., $\sec^2 y\frac{dy}{dx} + (2\tan y)x = x^3$ ... (1)

To reduce (1) in linear differential equation, we substitute $z = \tan y$

so that $\frac{dz}{dx} = \sec^2 y\frac{dy}{dx}$

Then (1) reduces to $\frac{dz}{dx} + 2zx = x^3$ ... (2)

Equation (2) is a linear differential equation. So the I.F. $= e^{\int 2x\, dx} = e^{x^2}$

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Multiplying (2) with the I.F. e^{x^2} and integrating,

$\int d(ze^{x^2}) = \int x^3 e^{x^2} dx + c$, c being an arbitrary constant

i.e., $ze^{x^2} = \frac{1}{2}\int ue^u dx + c$ [To evaluate R.H.S. we put $u = x^2$]

i.e., $\tan y\, e^{x^2} = \frac{1}{2}(ue^u - e^u) + c$

i.e., $(\tan y)e^{x^2} = \frac{1}{2}(x^2 - 1)e^{x^2} + c$

i.e., $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

Which is the required general solution for c being an arbitrary constant.

6. **Reduce the equation $\sin y \frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$ to a linear equati and hence solve it.** [CP 201

Solution Here the given differential equation is

$\sin y \frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$

i.e., $\sin y \frac{dy}{dx} - 2\cos y \cos x = -\cos x \sin^2 x$ ... (1)

To reduce (1) in linear form, we substitute $z = -\cos y$

Then, $\frac{dz}{dx} = \sin y \frac{dy}{dx}$. With these transformations (1) reduces to

$$\frac{dz}{dx} + 2z\cos x = -\cos x \sin^2 x \quad \text{... (2)}$$

Equation (2) is a first order linear differential equation.

Therefore the I.F. of (1) is $e^{\int 2\cos x dx} = e^{2\sin x}$

Multiplying (2) by the I.F. $e^{2\sin x}$ and then integrating, we have

$(ze^{2\sin x}) = -\int \cos x \sin^2 x\, e^{2\sin x} dx + c$, c being and arbitrary constant

To evaluate the integration in R.H.S., we substitute, $z = \sin x$,

then $\cos x dx = dz$ and we have

$$\int \cos x \sin^2 x\, e^{2\sin x} dx = \int z^2 e^{2z} dz = \frac{z^2 e^{2z}}{2} - \int z e^{2z} dz$$

$$= \frac{z^2 e^{2z}}{2} - \frac{ze^{2z}}{2} + \frac{e^{2z}}{4} = \frac{1}{4}(e^{2\sin x} - 2\sin x e^{2\sin x} + 2\sin^2 x e^{2\sin x})$$

Therefore the general solution is

$$ze^{2\sin x} = \frac{1}{4}(e^{2\sin x} - 2\sin x e^{2\sin x} + 2\sin^2 x e^{2\sin x}) + c$$

i.e., $\cos y + \frac{1}{4}(2\sin^2 x - 2\sin x + 1) + ce^{-2\sin x} = 0$, where c is an arbitrary constant.

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7. Solve : $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x \sec y$

Solution Here the given differential equation is

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x \sec y \quad i.e., \cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1 + x)e^x \qquad \text{... (1)}$$

Let us substitute $z = \sin y$

Then $\frac{dz}{dx} = \cos y\frac{dy}{dx}$

We have from (1), $\frac{dz}{dx} - \frac{y}{1+x} = (1 + x)e^x$ ... (2)

Clearly differential equation (2) is a linear differential equation, therefore

I.F. $= e^{-\int\frac{dx}{1+x}} = e^{-\log(1+x)} = \frac{1}{1+x}$

Multiplying (2) by the I.F. $\frac{1}{1+x}$ and integrating we get $z\left(\frac{1}{1+x}\right) = \int(1 + x)e^x \cdot \frac{1}{1+x}dx + c$,

c being an arbitrary constant.

$i.e., \frac{\sin y}{1+x} = e^x + c$ $i.e.,$ $\sin y = (1 + x)(e^x + c)$, which is the required general solution.

8. Solve : $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$ [CP 1997, 2006, 2009, 2012; BP 1994]

Solution Here the given differential equation is

$$\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y} \quad i.e., \; y^{-\frac{1}{2}}\frac{dy}{dx} + y^{\frac{1}{2}}\frac{x}{1-x^2} = x \qquad \text{... (1)}$$

Clearly differential equation (1) is Bernoulli's form of differential equation. To reduce it in linear form we substitute $z = y^{\frac{1}{2}}$, so that, $\frac{dz}{dx} = \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$ $i.e.,$ $y^{-\frac{1}{2}}\frac{dy}{dx} = 2\frac{dz}{dx}$,

Then (1) becomes $2\frac{dz}{dx} + \frac{xz}{1-x^2} = x$ $i.e.,$ $\frac{dz}{dx} + \frac{x}{2(1-x^2)}z = \frac{x}{2}$ ... (2)

Clearly equation (2) is a linear differential equation.

So I.F. $= e^{\int\frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\log(x^2-1)} = (x^2-1)^{-\frac{1}{4}}$

Multiplying (2) by $(x^2-1)^{-\frac{1}{4}}$ and integrating we have,

$z(x^2-1)^{-\frac{1}{4}} = \int\frac{x}{2}(x^2-1)^{-\frac{1}{4}}dx + c$, c being an arbitrary constant

$= \frac{1}{4}\int u^{-\frac{1}{4}}du + c$, where $u = x^2 - 1$, put in R.H.S.

$= \frac{1}{4}\frac{u^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + c = -\frac{1}{3}(x^2-1)^{\frac{3}{4}} + c$ $i.e.,$ $\sqrt{y}\,(x^2-1)^{-\frac{1}{4}} + \frac{1}{3}(x^2-1)^{\frac{3}{4}} = c$

Which is the required general solution for c being an arbitrary constant.

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9. Solve : $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$ [CP 20..]

Solution Here the given differential equation is

$$\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y \quad i.e., \sec^2 y\frac{dy}{dx} + 2\frac{\tan y}{x} = x^3 \quad \dots (1)$$

To reduce (1) in linear differential equation, we put

$z = \tan y$, so that $\frac{dz}{dx} = \sec^2 y\frac{dy}{dx}$. Then (1) reduces to $\frac{dz}{dx} + \frac{2}{x}z = x^3$ ... (2)

Clearly (2) is a linear differential equation. So its I.F. $= e^{\int \frac{2}{x}dx} = x^2$.
Multiplying (2) by I.F. $= x^2$ and integrating we have,

$x^2 z = \int x^5 dx + c$, c being an arbitrary constant

i.e., $x^2 z = \frac{x^6}{6} + c$, *i.e.*, $6x^2 z = x^6 + c_1$, [$c_1 = 6c$, is another arbitrary constant.]

Which is the required general solution.

10. Solve : $\frac{dy}{dx} + \frac{y}{x} = y^2$ [CP 2013]

Solution Here the given differential equation is $\frac{dy}{dx} + \frac{y}{x} = y^2$ *i.e.*, $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1$ ... (1)

To reduce (1) into linear form, we substitute, $z = \frac{1}{y}$

Then $\frac{dz}{dx} = -\frac{1}{y^2}\cdot\frac{dy}{dx}$. Thus (1) reduces to $-\frac{dz}{dx} + \frac{z}{x} = 1$ *i.e.*, $\frac{dz}{dx} - \frac{z}{x} = -1$ ... (2)

Clearly (2) is a linear differential equation, with I.F. $= e^{-\int \frac{dx}{x}} = \frac{1}{x}$

Multiplying (2) by the I.F. $= \frac{1}{x}$ and then integrating,

$\frac{z}{x} = \int -\frac{dx}{x} + c$, where c is an arbitrary constant

i.e., $\frac{1}{xy} + \log\frac{1}{x} = c$, which is the required general solution.

11. Solve : $\frac{dy}{dx} + (x\tan^{-1}y - x^3)(1 + y^2) = 0$, when $x = 0, y = 1$ [CP 2013]

Solution Here the given differential equation is

$\frac{dy}{dx} + (x\tan^{-1}y - x^3)(1 + y^2) = 0$ *i.e.*, $\frac{1}{1+y^2}\frac{dy}{dx} + x\tan^{-1}y = x^3$

To solve (1), we put $z = \tan^{-1}y$, so that $\frac{dz}{dx} = \frac{1}{1+y^2}\frac{dy}{dx}$ ... (1)

Then (1) reduces to $\frac{dz}{dx} + xz = x^3$

... (2)

Clearly (2) is a first order linear differential equation. Therefore its I.F. = $e^{\int x\,dx} = e^{x^2/2}$

Multiplying (2) by the I.F. $e^{x^2/2}$ and integrating we have,

$ze^{x^2/2} = \int x^3 e^{x^2/2}\,dx + c$, c being an arbitrary constant ... (3)

To evaluate $\int x^3 e^{x^2/2}\,dx$, we put $u = \frac{x^2}{2}$ so that $2du = x\,dx$.

Then $\int x^3 e^{x^2/2}\,dx = \int 2ue^u (2du)$

$$= 4\int ue^u du = 4(ue^u - e^u) = 4\left(\frac{x^2}{2}e^{x^2/2} - e^{x^2/2}\right) = 2(x^2-2)e^{x^2/2}$$

Then from (3) we get, $ze^{x^2/2} = 2(x^2-2)e^{x^2/2} + c$

i.e., $\tan^{-1}y = 2(x^2-2) + ce^{-x^2/2}$

which is the required general solution for c being an arbitrary constant

Using the initial condition $x = 0$, when $y = 1$, we have

$\tan^{-1}(1) = 2(-2) + c$ *i.e.*, $c = \frac{\pi}{4} + 4$

Hence the required particular solution is $\tan^{-1}(y) = 2(x^2-2) + \left(\frac{\pi}{4} + 4\right)e^{-x^2/2}$

12. Solve : $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$ [CP 2012]

Solution Here the given differential equation is

$\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$ *i.e.*, $\frac{1}{y(\log y)^2}\cdot\frac{dy}{dx} + \frac{1}{x\log y} = \frac{1}{x^2}$ ... (1)

To reduce (1) in linear form, we put $z = \frac{1}{\log y}$

Then $\frac{dz}{dx} = -\frac{1}{(\log y)^2}\cdot\frac{1}{y}\cdot\frac{dy}{dx}$

Therefore (1) reduces to, $-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$ *i.e.*, $\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$ ... (2)

Clearly (2) is a first order linear differential equation whose I.F. = $e^{\int -1/x\,dx} = e^{-\log x} = \frac{1}{x}$

Multiplying (2) by I.F. $\frac{1}{x}$ and integrating, we have

$\frac{z}{x} = -\int \frac{1}{x^3}dx + c$, c being an arbitrary constant

i.e., $\frac{1}{x\log y} = \frac{1}{2x^2} + c$, which is the required general solution.

13. Solve : $x\,dy - y\,dx - \cos\frac{1}{x}dx = 0$ [CP 2008]

Solution Here the given differential equation is $x\,dy - y\,dx - \cos\frac{1}{x}dx = 0$

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i.e., $x\frac{dy}{dx} - y = \cos\frac{1}{x}$ i.e., $\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x}\cos\frac{1}{x}$ ... (1)

Clearly (1) is a first order linear differential equation

Therefore I.F. = $e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

Multiplying (1) by the I.F. $\frac{1}{x}$ and then integrating we have,

$\frac{y}{x} = \int \frac{1}{x^2}\cos\frac{1}{x}\,dx + c$, c is an arbitrary constant

$= -\int d\left(\sin\frac{1}{x}\right) + c = c - \sin\frac{1}{x}$

i.e., $\frac{y}{x} + \sin\frac{1}{x} = c$, which is the required general solution for c being an arbitrary constant.

14. **Find the integrating factor of $\cos x\frac{dy}{dx} + y\sin x = 1$** [CP 2007]

Solution Here the given differential equation is $\cos x\frac{dy}{dx} + y\sin x = 1$

i.e., $\frac{dy}{dx} + y\tan x = \sec x$ ... (1)

Clearly (1) is a first order linear differential equation.

Therefore its integrating factor is $e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$.

15. **Solve : $(x\cos x)\frac{dy}{dx} + y(\sin x + \cos y) = 1$** [CP 2007]

Solution Here the given differential equation is

$(x\cos x)\frac{dy}{dx} + y(x\sin x + \cos y) = 1$ i.e., $\frac{dy}{dx} + (\tan x + \frac{1}{x})y = \frac{1}{x\cos x}$ ... (1)

Clearly (1) is a first order linear differential equation.

Therefore the I.F. = $e^{\int(\tan x + \frac{1}{x})dx} = e^{\log\sec x + \log x} = x\sec x$

Multiplying (1) by the I.F. $x\sec x$ and then integrating we have the required general solution $(x\sec x)y = \int \sec^2 x\,dx + c$, c being an arbitrary constant

i.e., $xy\sec x = \tan x + c$ i.e., $xy = \sin x + c\cos x$.

16. **Solve : $x\frac{dy}{dx} + y\log y = xye^x$** [CP 2006]

Solution Here the given differential equation is $x\frac{dy}{dx} + y\log y = xye^x$

i.e., $\frac{1}{y}\cdot\frac{dy}{dx} + \frac{\log y}{x} = e^x$ ... (1)

The equation (1) can be reduced into linear form by putting

$z = \log y$. Then $\frac{dz}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$.

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With this transformation, (1) reduces to $\frac{dz}{dx}+\frac{z}{x}=e^x$ ... (2)

Clearly (2) is first order linear differential equation. So its I.F. = $e^{\int dx/x}=e^{\log x}=x$

Multiplying (2) by the I.F. = x and then integrating, the general solution of the given equation is

$zx=\int xe^x+c$, where c is an arbitrary constant

i.e., $x\log y=xe^x-e^x+c$.

Exercise

1. Reduce the equation $\frac{dy}{dx}+f(x)\cdot y=g(x)\cdot y^n$ to linear form and find the general solution. **[CP 1999]**
2. Arrange the differential equation $(1+y^2)dx-(\tan^{-1}y-x)dy=0$ as a linear form.
3. Convert the equation $x\frac{dy}{dx}+y=y^2\log x$ into a linear equation.
4. Reduce the equation $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$ into the form $\frac{dy}{dx}+py=0$ **[CP 1987]**
5. Convert the equation $y(2xy+e^x)dx-e^xdy=0$ into a linear equation.
6. Find the integrating factor of the differential equation $\cos^2x\frac{dy}{dx}+y=\tan x$. **[CP 2001, 2008, 2011]**
7. Find the integrating factor of the equation $(1+y^2)+(x-e^{-\tan y})\frac{dy}{dx}=0$ **[VP 2002]**
8. Find the integrating factor of the equation $\frac{dy}{dx}+\frac{x}{1+y^2}=\frac{e^{-\tan^{-1}y}}{1+y^2}$
9. Find the I.F. of the equation $\frac{dy}{dx}+\frac{x}{2(1-x^2)}y=\frac{x}{2}$
10. Find the I.F. of the differential equation $x\log x\cdot\frac{dy}{dx}+y=2\log x$

- **Solve the following differential equations (11-21):**

11. $\frac{dy}{dx}+2xy=xy^3$ **[CP 1991]**
12. $(x+2y^3)\frac{dy}{dx}=y$
13. $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$ **[BP 1995]**
14. $(1-x^2)\frac{dy}{dx}-2xy=x-x^3\ (x^2\neq 1)$ **[CP 1997]**
15. $(x+y+1)dy=dx$ **[BP 1990]**
16. $x\frac{dy}{dx}-ay=x+1$
17. $x\log x\frac{dy}{dx}+y=2\log x$ **[BP 1994]**
18. $y^2\frac{dy}{dx}-2\frac{y^3}{x}=2x^2$ given that $y=1$ when $x=1$
19. $(x^2y^3+2xy)dy=dx$
20. $\frac{dy}{dx}=\frac{y}{x+\sqrt{xy}}$
21. $\frac{dy}{dx}=(4e^{-y}\sin x-1)$
22. Find the equation of a curve whose slope at any point is equal to $(x+2y+1)/x$ and which passes through the fixed point (1, 0).

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23. Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y = 2(e^x - x - 1)$ [VP 2002]

24. Show that the equation of the curve whose slope at any point is equal to $\frac{1}{x+y+1}$ and passes through the point (1, 0) is $x + y + 2 = 3e^y$

Answers

1. **Linear form :**
$$\frac{du}{dx} + (1-n)f(x)u = (1-n)g(x),$$
where $u = y^{1-n}$

General solution :
$$y^{1-n}e^{(1-n)\int f(x)dx} = (1-n)\int g(x)e^{(1-n)\int f(x)dx}dx + c$$

2. $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

3. $\frac{dz}{dx} - z\cdot\frac{1}{x} = -\frac{\log x}{x}$, when $z = \frac{1}{y}$

4. $\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$, when $z = \frac{1}{y}$

5. $\frac{dz}{dx} + z = -2xe^{-x}$, when $z = \frac{1}{y}$

6. $e^{\tan x}$

7. $e^{\tan^{-1}y}$

8. $e^{\tan^{-1}y}$

9. $(1-x^2)^{-1/4}$

10. $\log x$

11. $\frac{2}{y^2} = 1 + 2e^{-2x^2}$

12. $x = y^3 + cy$

13. $\frac{x}{y} = \frac{1}{2} + cx^2$

14. $4y(1-x^2) = x^2(2-x^2) + c$

15. $x + y + 2 = ce^y$

16. $y + \frac{1}{a} = \frac{x}{1-a} + ce^a$

17. $y\log x = c + (\log x)^2$

18. $y^3 + 2x^3 = 3x^6$

19. $\frac{2}{x} + y^2 = 1 + ce^{-y^2}$

20. $\sqrt{\frac{x}{3}} = \log\sqrt{y} + c$

21. $e^{x+y} = 2e^x(\sin x - \cos x) + c$

22. $y = (3x^2 - 2x - 1)/2$

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5 CHAPTER

First Order but not of First Degree Differential Equation

Introduction

In this chapter (*i.e.* differential equation with first order but not of first degree) we denote the differential of y (*i.e.* dependent variable) with respect to x (independent variable) by $p\left(=\frac{dy}{dx}\right)$. Therefore the differential equation of the form $(x, y, p) = 0$ is a first order differential equation but may not be of first degree and in this chapter we describe several approaches to solve these types of differential equation.

5.1 Equations Solvable for p

We consider a first order and n degree differential equation as follows :

$$p^n + a_1p^{n-1} + a_2p^{n-2} + \dots + a_{n-1}p_1 + a_n = 0 \quad \dots (1)$$

where a_i $(i = 1, 2, \dots, n)$ are functions of x and y.

Now the L.H.S. of (1) and easily be factorized in n linear factors in p. Then (1) can be re-written as $(p - \phi_1(x, y))(p - \phi_2(x, y)) \dots (p - \phi_n(x, y)) = 0$ ... (2)

where $\phi_1, \phi_2, \dots, \phi_n$ are function of x and y.

Thus either any factor or more than one linear factors of L.H.S. of (2) should vanish to obtain the solution from (2). However this is true if any one linear factor of L.H.S. of (2) becomes zero or any arbitrary combination of linear factors become zero. Now we can easily solve the first order first degree differential equations $p - \phi_i(x, y) = 0$; $i = 1, 2, \dots, n$.

Let the solution for each of $p - \phi_i(x, y) = 0$; $i = 1, 2, \dots, n$ is $\psi_i(x, y, c_i) = 0$; $i = 1, 2, \dots, n$ and $c_1, c_2, \dots, c_n$ are n arbitrary constants. Therefore the required general solution will be $\psi_1(x, y, c)\psi_2(x, y, c) \dots \psi_n(x, y, c) = 0$, where c is the arbitrary constant replacing all of $c_1, c_2, \dots, c_n$; since for first order differential equation there will be only one arbitrary constant.

5.2 Equations Solvable for y

Let us consider a first order differential equation as follows :

$$y = f(x, p) \quad \dots (3)$$

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Any first order differential equation of the form (3) can be solved for the dependent variable y.

If we differentiate (3) w.r. to x, then (3) becomes $p = \frac{d}{dx}\{f(x, p)\} = g\left(x, p, \frac{dp}{dx}\right)$ ... (4)

The equation (4) is not directly involved with the variable y and equation (4) can be solved easily for p and x. Let, after solving solution of (4) becomes $F(x, p, c) = 0$ ... (5)

where c is an arbitrary constant.

Hence the general solution of the given differential equation is the solution which can be obtained from eliminating p from both (3) and (5).

5.3 Equations Solvable for x

Let us consider a first order differential equation as follows :

$$x = f_1(y, p) \quad \left[p = \frac{dy}{dx}\right] \qquad \text{... (6)}$$

Any first order differential equation whose form is (6), can be solved for the independent variable x. Differentiating (6) w.r. to y we have,

$$\frac{1}{p} = \frac{d}{dy}\{f_1(y, p)\} = g_1\left(y, p, \frac{dp}{dy}\right) \qquad \text{... (7)}$$

clearly the equation (7) does not correspond with x and the differential equation (7) can be solved for p and y easily.

Let the general solution of the differential equation (7) be $F_1(y, p, c) = 0$ where c is an arbitrary constant. ... (8)

Hence the required general solution of the given differential equation (6) can be obtained by eliminating p from the given equation and the solution (i.e. eliminating p from $x = f_1(y, p)$ and $F_1(y, p, c) = 0$).

5.4 Some Special Cases

Let us consider the quations containing no x or no y. We consider a first order differential equation where there is no independent variable as follows : $f_2(y, p) = 0 \left[p = \frac{dy}{dx}\right]$ ... (8)

If (8) can be solvable for p, then we have from (8), $p = \frac{dy}{dx} = g_2(y)$ and the general solution becomes $x + c = \int \frac{dy}{g_2(y)}$, c being an arbitrary constant.

Again if (8) can be solvable for y, then we can write (8) as $y = g_3(p)$, and this equation can easily be solved by differentiating with respect to x and then integrating (which already have been discussed).

On the other hand let a first order differential equation free from dependent variable y i.e., it is in the form $f_3(x, p) = 0 \quad \left[p = \frac{dy}{dx}\right]$ ... (9)

Like earlier case, if equation (9) is solvable for p, then we have

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$p = \frac{dy}{dx} = g_4(x)$, where general solution is $y + c = \int g_4(x)dx$, c being an arbitrary constant

However if the equation (9) is solvable for x, then we can write from (9), $x = g_5(p)$, and this differential equation can again easily be solved by differentiating with respect to y and then integrating (which aleady have been discussed earlier).

Note

> In general the differential equation which is a polynomial function of p of order n can be solved by solvable for p method.

Illustrative Examples

1. **Solve : $xy(p^2 - 1) = (x^2 - y^2)p$** [CP 2014]

Solution Here the given differential equation is $xy(p^2 - 1) = (x^2 - y^2)p$ ... (1)

The equation (1) is quadratic in p and therefore it is solvable for p. So we have from (1)

$xyp^2 - px^2 - xy + py^2 = 0$ *i.e.*, $px(py - x) - y(py - x) = 0$

i.e., $(py - x)(px - y) = 0$ ... (2)

So from (2) we have either $py - x = 0$ or $px - y = 0$

when $py - x = 0$, then $y\frac{dy}{dx} - x = 0$ *i.e.*, $2ydy - 2xdx = 0$

Integrating we have, $y^2 - x^2 + c_1 = 0$, c_1 is an arbitrary constant.

Again when $px - y = 0$, then $x\frac{dy}{dx} - y = 0$ i.e. $\frac{dy}{y} - \frac{dx}{x} = 0$

Integrating we have $\log\frac{y}{x} = \log c_2$, c_2 is an arbitrary constant

i.e., $y - c_2x = 0$

Hence the required general solution is $(y^2 - x^2 + c)(y - cx) = 0$,

where $c(= c_1 = c_2)$ is an arbitrary constant.

2. **Solve : $p^2 - p(e^x + e^{-x}) + 1 = 0$**

Solution Here the given differential equation is $p^2 - p(e^x + e^{-x}) + 1 = 0$

i.e., $(p - e^x)(p - e^{-x}) = 0$ ... (1)

From (1) either $p - e^x = 0$ or $p - e^{-x} = 0$

when $p - e^x = 0$ then $\frac{dy}{dx} - e^x = 0$ *i.e.*, $dy - e^xdx = 0$

Now integrating, we have $y - e^x + c_1 = 0$, c_1 is an arbitrary constant

Again, when $p - e^{-x} = 0$ then $\frac{dy}{dx} - e^{-x} = 0$, *i.e.*, $dy - e^{-x}dx = 0$

Now integrating, we have $y + e^{-x} + c_2 = 0$, c_2 is an arbitrary constant

Hence the required general solution is $(y - e^x + c)(y + e^{-x} + c) = 0$.

where $c(= c_1 = c_2)$ is an arbitrary constants.

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3. **Solve : $p^2 + px = xy + y^2$** [CP 1996]

Solution $p^2 + px = xy + y^2$ *i.e.*, $p^2 - y^2 + px - xy = 0$

i.e., $(p - y)(p + y) + x(p - y) = 0$ *i.e.*, $(p - y)(p + x + y) = 0$

From (1) either $p - y = 0$ or $p + x + y = 0$ ... (1)

when $p - y = 0$ then $\frac{dy}{dx} - y = 0$ *i.e.*, $\frac{dy}{y} - dx = 0$

Now integrating, we have $\log y - x + c_1 = 0$, c_1 being an arbitrary constant ... (2)

Again, when $p + x + y = 0$ then $\frac{dy}{dx} + y = -x$

Thus I.F. of (3) $= e^{\int dx} = e^x$ ... (3)

Now multiplying both sides of (3) by e^x and then integrating, we get $y \cdot e^x = -\int xe^x dx + c_2$

i.e., $ye^x = -[xe^x - e^x] + c_2$ *i.e.*, $(x + y - 1)e^x - c_2 = 0$, c_2 being an arbitrary constant

Hence the required general solution is $(\log y - x + c)[(x + y - 1)e^x - c] = 0$, $c\ (= c_1 = c_2)$ is an arbitrary constant.

4. **Solve : $p^3 - (x^2 + xy + y^2)p^2 + (x^3y + x^2y^2 + xy^3)p - x^3y^3 = 0$** [MH 1992]

Solution Here the given differential equation is

$$p^3 - (x^2 + xy + y^2)p^2 + (x^3y + x^2y^2 + xy^3)p - x^3y^3 = 0$$

i.e., $(p - x^2)(p - y^2)(p - xy) = 0$, $p = \frac{dy}{dx}$

So from (1) we have either $p - x^2 = 0$, or $p - y^2 = 0$ or, $p - xy = 0$ ... (1)

when $p - x^2 = 0$, then $\frac{dy}{dx} - x^2 = 0$ *i.e.*, $dy - x^2dx = 0$ and on integration we have

$y - \frac{x^3}{3} + \frac{c_1}{3} = 0$ *i.e.*, $3y - x^3 + c_1 = 0$, c_1 being an arbitrary constant

when $p - y^2 = 0$, then $\frac{dy}{dx} - y^2 = 0$ *i.e.*, $y^{-2}dy - dx = 0$. On integration we have

$-\frac{1}{y} - x - c_2 = 0$ *i.e.*, $xy + yc_2 + 1 = 0$, c_2 being an arbitrary constant.

Lastly, when $p - xy = 0$ then $\frac{dy}{dx} - xy = 0$ *i.e.*, $2x\,dx - 2\frac{dy}{y} = 0$

Integrating, we have $x^2 - \log c_3y^2 = 0$, c_3 being an arbitrary constant.

On taking the arbitrary constant as c, the required general solution of the given differential equation thus becomes $(3y - x^3 + c)(xy + cy + 1)(x^2 - \log cy^2) = 0$.

5. **Solve : $y = p\tan p + \log\cos p$**

Solution Here the given differential equations is $y = p\tan p + \log\cos p$ ... (1)

clearly the differential equation (1) is of first order but not of first degree and does not contain x. Thus it is solvable for y.

Differentiating (1) w.r. to x, we have $p = p\sec^2 p\frac{dp}{dx} + \tan p\frac{dp}{dx} - \tan p\frac{dp}{dx}$, $p = \frac{dy}{dx}$

i.e., $p\sec^2 p\frac{dp}{dx} = p$, *i.e.*, $dx = \sec^2 p\,dp$

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Integrating we get, $x + c = \tan p$, c being an arbitrary constant.

Since, $\cos p = \frac{1}{\sqrt{\sec^2 p}} = \frac{1}{\sqrt{1+\tan^2 p}}$, then eliminating p from (1) and $x + c = \tan p$,

the required general solution is $y = (x + c)\tan^{-1}(x + c) + \log\left(\frac{1}{\sqrt{1+(x+c)^2}}\right)$

i.e., $y = (x + c)\tan^{-1}(x + c) - \frac{1}{2}\log(1 + (x - c)^2)$

6. **Solve : $x = 4p + 4p^3$** [CH 1982]

Solution Here the given differential equation is $x = 4p + 4p^3$ ... (1)

The equation (1) is solvable for x.

Differentiating (1) w.r. to y, we get, $\frac{1}{p} = (4 + 12p^2)\frac{dp}{dy}$, $p = \frac{dy}{dx}$ i.e., $dy = 4(p + 3p^3)dp$

Integrating both sides we have, $y + c = 2p^2 + 3p^4$, ... (2)
c being an arbitrary constant.

Eliminating p between (1) and (2) we have the required general solution of the given differential equation.

7. **Solve : $y = x + a\tan^{-1}p$**

Solution Here the given differential equation is $y = x + a\tan^{-1}p$ ... (1)

Clearly the differential equation (1) is solvable for y.

Differentiating (1) w.r. to x, we have, $p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$, $p = \frac{dy}{dx}$

i.e., $p - 1 = \frac{a}{(p^2+1)}\frac{dp}{dx}$ i.e., $\frac{dx}{a} = \frac{dp}{(p^2+1)(p-1)}$ ... (2)

Let, $\frac{1}{(p^2+1)(p-1)} = \frac{Ap+B}{p^2+1} + \frac{C}{p-1} = \frac{(A+C)p^2+(B-A)p+C-B}{(p^2+1)(p-1)}$

Therefore, $A + C = 0$, $B - A = 0$, $C - B = 1$ i.e., $A = B = -C = -\frac{1}{2}$

Therefore, $\frac{1}{(p^2+1)(p-1)} = -\frac{1}{2}\frac{p+1}{p^2+1} + \frac{1}{2}\frac{1}{p-1}$

Then from (2), we have $\frac{dx}{a} = -\frac{1}{2}\frac{pdp}{p^2+1} - \frac{1}{2}\frac{dp}{p^2+1} + \frac{1}{2}\frac{dp}{p-1}$

Integrating we have

$x + c = a\left[-\frac{1}{4}\log(p^2 + 1) + \frac{1}{2}\log(p - 1) - \frac{1}{2}\tan^{-1}p\right]$, c being an arbitrary constant

i.e., $x + c = \frac{a}{2}\left[\log\left((p - 1)(p^2+1)^{-\frac{1}{2}}\right) - \tan^{-1}p\right]$ ... (3)

Hence the required general solution is obtained for eliminating p form the the equations (1) and (3).

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8. **Solve : $y = p^2x + p$** [CH 19

Solution Here the given differential equation is $y = p^2x + p$

Clearly the equation (1) is solvable for y. ... (1)

So differentiating (1) w.r. to x, we have $p = p^2 + 2px\frac{dp}{dx} + \frac{dp}{dx}$

i.e., $(1 + 2px)\frac{dp}{dx} = p - p^2$ i.e., $(p^2 - p)dx + (1 + 2px)dp = 0$ ... (2)

Comparing (2) with $Mdx + Ndp = 0$, we have $M = p^2 - p$ and $N = 1 + 2px$

Now $\frac{\partial M}{\partial p} = 2p - 1$ and $\frac{\partial N}{\partial x} = 2p$, therefore (2) is not an exact differential equation.

Now $\frac{1}{M}\left(\frac{\partial M}{\partial p} - \frac{\partial N}{\partial x}\right) = \frac{1}{p^2 - p}(2p - 1 - 2p) = \frac{-1}{p^2 - p} = \phi(p)$, [say]

Therefore one I.F. of (2) is $e^{\int -\phi(p)dp} = e^{\int \frac{dp}{p^2 - p}} = e^{-\int \frac{dp}{p} + \int \frac{dp}{p-1}} = \frac{p-1}{p}$

Multiplying (2) by the I.F $\left(\frac{p-1}{p}\right)$ we have, $(p - 1)^2dx + \frac{(p-1)(1+2px)}{p}dp = 0$ ... (3)

Now $\int (p - 1)^2 dx = (p - 1)^2 x$ [Assuming p as constant]

and $\int \frac{(p-1)(1+2px)}{p}dp = \int (1 + 2px)dp - \int \frac{1+2px}{p}dp = p + p^2x - 2xp - \log p$

$= (p - 1)^2 x - \log p + p - x$

Combining, the solution of (2) is $(p - 1)^2 x + p = x + \log p + c$,

c being an arbitrary constant. ... (4)

Hence the required general solution can be obtained by eliminating p from (1) and (4).

9. **Solve : $y = p\cos p - \sin p$**

Solution Here the given equation is $y = p\cos p - \sin p$

Differentiating (1) w.r. to x (both sides) $\frac{dy}{dx} = (\cos p - p\sin p)\frac{dp}{dx} - \cos p\frac{dp}{dx}$ ... (1)

i.e., $p = (\cos p - p\sin p - \cos p)\frac{dp}{dx}$ i.e., $p = -p\sin p\frac{dp}{dx}$ i.e., $-\sin p\, dp = dx$

Now integrating, $\int dx = -\int \sin p\, dp + c$ giving $x = \cos p + c$, c being an arbitrary constant.

Hence the required solution is $x = \cos p + c; y = p\cos p - \sin p$, c being an arbitrary constant.

10. **Solve : $p^2 - 2xp + 1 = 0$**

Solution Here the given equation is $p^2 - 2xp + 1 = 0$

Differentiating (1) w.r. to x, (both sides) we have, $2p\frac{dp}{dx} - 2p - 2x\frac{dp}{dx} = 0$ ... (1)

i.e., $(p - x)\frac{dp}{dx} = p$ i.e., $\left(p - \frac{p^2+1}{2p}\right)\frac{dp}{dy}\cdot\frac{dy}{dx} = p$

i.e., $\frac{p^2-1}{2p}\cdot\frac{dp}{dy}\cdot p = p$ i.e., $\frac{p^2-1}{2p}dp = dy$

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Now integrating, we get $\int dy = \frac{1}{2}\int p\,dp - \frac{1}{2}\int \frac{1}{p}\,dp + c$

i.e., $y = \frac{p^2}{4} - \frac{1}{2}\log p + c$, c is an arbitrary constant. ... (2)

Hence the general solution of the given equation can be obtained by eliminating p from (1) and (2), i.e., eliminating p from the equations $x = \frac{p^2+1}{2p}$; $y = \frac{p^2}{4} - \frac{1}{2}\log p + c$.

Exercise

1. Solve : $p^2 + 2px - 3x^2 = 0$ [CH 1987]
2. Solve : $xp^2 + (y - x)p - y = 0$ [TH 1991]
3. Solve : $p^2 + p - 6 = 0$ [CP 1990]
4. Solve : $p^2y - p(xy + 1) + x = 0$
5. Solve : $p^2 - (x + y)p + xy = 0$ [CP 1982]
6. Solve : $p^2 + 2px + py + 2xy = 0$
7. Solve : $xy\left\{\left(\frac{dy}{dx}\right)^2 - 1\right\} = (x^2 - y^2)\frac{dy}{dx}$ [CH 1985]

   [*Hint* : Take $p = \frac{dy}{dx}$ and then solve for p.]
8. Solve : $y\left\{x(2x + 1)\frac{dy}{dx} - y\left(\frac{dy}{dx}\right)^2\right\} = 2x^3$ [CH 1986]
9. Solve : $y - (1 + p^2)^{-\frac{1}{2}} - b = 0$
10. Solve : $y = 2px + p^2$ [VH 1997]
11. Solve : $e^y - p^3 - p = 0$ [CH 1982]
12. Solve : $y = Ap^2 + Bp^3$ [CP 1992, 2005]
13. Solve : $x = py - p^2$

    [*Hint* : The equation is solvable for x.]
14. Solve : $x + \frac{p}{\sqrt{1 + p^2}} = a$
15. Solve : $x = y + a\log p$
16. Solve : $p^2 + 2py\cot x = y^2$
17. Solve : $y = 2px + y^2p^3$ [CP 1995]

Answers

1. $(2y - x^2 + c)(2y + 3x^2 + c) = 0$
2. $(xy + c)(y - x + c) = 0$
3. $(y + 3x - c)(y - 2x - c) = 0$
4. $(x^2 - 2y - c)(2x - y^2 - c) = 0$
5. $(2y - x^2 - c)(\log y - c) = 0$
6. $(x^2 + y - c)(\log y + x - c) = 0$
7. $(xy - c)(x^2 - y^2 - 2c) = 0$
8. $(x^2 - y^2 + 2c)(4x^3 - 3y^2 + 6c) = 0$
9. $(x + c)^2 + (y + b)^2 = 1$
10. $4(x^2 + y)^3 = (3xy + 2x^3 + c)^2$
11. Eliminating p between $y = \log(p + p^3)$ and $x = 2\tan^{-1}p - p^{-1} + c$
12. Eliminating p between $y = Ap^2 + Bp^3$ and $x = 2Ap + \frac{3}{2}Bp^2 + c$
13. Eliminating p between $x = py - p^2$ and $y = p + (c + \cosh^{-1}p)(p^2 - 1)^{-\frac{1}{2}}$
14. $(x - a)^2 + (y + c)^2 = 1$
15. $y = c - a\log(e^{(x-y)/a} - 1)$
16. $y(1 \pm \cos x) = c$
17. $y^2 = 2cx + c^3$

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6 CHAPTER

Differential Equation of Clairaut's Form

Introduction

In some special cases, one or some solutions of a differential equation can never be obtained from its general solution. In such cases, these types of solutions are known as singular solution. Among various types of differential equation, for first order differential equation, Clairaut's type differential equation is an example of that type of differential equation, which not only obtained solutions for higher degree differential equations but also possess singular solution.

6.1 Clairaut's form : Its General and Singular Solution

The first order differential equation of the form $y = px + f(p)$, $p = \frac{dy}{dx}$ is known as Clairaut's equation. ... (1)

To solve the differential equation (1), we first differentiate (1) w.r. to x and we obtain,

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad i.e., \ (x + f'(p))\frac{dp}{dx} = 0 \qquad \text{... (2)}$$

From (2) we have either $\frac{dp}{dx} = 0$ or $x + f'(p) = 0$

When $\frac{dp}{dx} = 0$, then $dp = 0$ and on integrating $p = c$, c being an arbitrary constant. Thus the general solution for the given differential equation is $y = cx + f(c)$ ... (3)

Next to find the singular solution, we make the other factor [of the equation (2)] as zero and we have, $x + f'(p) = 0$ ... (4)

Eliminating p between (1) and (4), we have the required singular solution.

Note

The singular solution has no arbitrary constant(s). Even this singular solution can not be determined from the general solution.

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6.2 Alternative Method to find Singular Solution of Clairaut's Form

Suppose the differential equation for Clairaut's form is given by

$$f(x, y, p) = Ap^2 + Bp + C \quad \text{... (5)}$$

where $p = \frac{dy}{dx}$ and A, B, C are functions of both x and y. Then like earlier, the general solution of (5) would be $f(x, y, c) = Ac^2 + Bc + C$, where C is an arbitrary constant.

However the singular solution can be obtained by using the condition for having two equal roots of the equation $f(x, y, p) = 0$. Thus the singular solution can be obtained from

$$B^2(x, y) - 4A(x, y)C(x, y) = 0 \quad \text{... (6)}$$

6.3 Equation Reducible to Clairaut's Equation

Let us consider a differential equation of first order as follows :

$$F(x, y, p) = 0, p = \frac{dy}{dx} \quad \text{... (7)}$$

If the first order differential equation (7) cannot be put in the form of Clairaut's equation, still with suitable transformations, equation (7) may be transformed to a Clairaut's differential equation. However, it should be noted that each differential equation of the form (7) cannot be reduced in Clairaut's form.

Note

Singular solution of the Clairaut's equation is directly related to the envelope of the family of a curve. Each and every member of the family of lines determined by the general solution of problem is tangent of the corresponding singular solution.

6.4 Extended form of Clairaut's Equation

Consider a differential equation of the form $y = xf_1(p) + f_2(p), p = \frac{dy}{dx}$ ... (8)

To solve (8), differentiating with respect to x, we have

$$p = f_1(p) + (xf_1'(p) + f_2'(p))\frac{dp}{dx}$$

$$\text{i.e., } \frac{dx}{dp} + x\frac{f_1'(p)}{f_1(p) - p} = \frac{f_2'(p)}{p - f_1(p)} \quad \text{... (9)}$$

Clearly (9) is a linear (on x) first order differential equation and let its general solution be

$$\phi(x, p, c) = 0 \quad \text{... (10)}$$

where c is an arbitrary constant. Thus eliminating p between (8) and (10), we have the required solution.

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Illustrative Examples

1. **Obtain the general and singular solution of $y = px + \sqrt{4p^2+9}$**

[CP 2008, 2010, 2016]

Solution Here the given equation is $y = px + \sqrt{4p^2+9}$ ... (1)

The equation (1) is in Clairaut's form. Differentiating (1) with respect to x, we have

$$p = p + x\frac{dp}{dx} + \frac{4p}{\sqrt{4p^2+9}}\frac{dp}{dx}, \left[p = \frac{dy}{dx}\right] \text{ i.e., } \left(x + \frac{4p}{\sqrt{4p^2+9}}\right)\frac{dp}{dx} = 0 \quad ... (2)$$

So from (2) we have, either $\frac{dp}{dx} = 0$ or $x + \frac{4p}{\sqrt{4p^2+9}} = 0$

Now $\frac{dp}{dx} = 0$ gives $dp = 0$ *i.e.*, $p = c$, c being an arbitrary constant.

Substituting $p = c$ in (1) we have the required general solution, $y = cx + \sqrt{4c^2+9}$

Again $x + \frac{4p}{\sqrt{4p^2+9}} = 0$ gives $x = -\frac{4p}{\sqrt{4p^2+9}}$ ... (3)

From (1) and (3), we have $y = -\frac{4p^2}{\sqrt{4p^2+9}} + \sqrt{4p^2+9}$

$$\text{i.e., } y = \frac{-4p^2+4p^2+9}{\sqrt{4p^2+9}} = \frac{9}{\sqrt{4p^2+9}} \quad ... (4)$$

From (3) and (4), we have $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \frac{4p^2+9}{4p^2+9} = 1$ *i.e.*, $\frac{x^2}{4} + \frac{y^2}{9} = 1.$

Which is the required singular solution of the given differential equation.

Note

The differential equation of the form $y = px + \sqrt{a^2p^2+b^2}$ has the singular solution $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

2. **Find the general solution and the singular solution of $y = px + p - p^2$.**

[CP 2007, 2013]

Solution Here the given differential equation is $y = px + p - p^2$ ... (1)

The equation (1) is in Clairaut's form. Differentiating equation (1) with respect to x, we have

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx} \left[p = \frac{dy}{dx}\right] \text{ i.e., } (x + 1 - 2p)\frac{dp}{dx} = 0 \quad ... (2)$$

So from (2) we have, either $\frac{dp}{dx} = 0$ or $x + 1 - 2p = 0$

Now, $\frac{dp}{dx} = 0$ gives $dp = 0$ integrating, $p = c$, c being an arbitrary constant

Substituting $p = c$ in (1), the required general solution is $y = cx + c - c^2$, c being an arbitrary constant.

Again, $x + 1 - 2p = 0$ gives $p = \frac{x+1}{2}$

Substituting $p = \frac{x+1}{2}$ in (1), the required singular solution is

$y = \frac{x+1}{2}x + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2$

i.e., $4y = 2(x^2 + x) + 2x + 2 - x^2 - 2x - 1 = x^2 + 2x + 1$ i.e., $(x + 1)^2 = 4y$

3. Obtain the general and singular solution of $y = px + ap(1 - p)$
[CH 1998, 2006, CP 1998, 2000, 2014]

Solution Here the given differential equation is $y = px + ap(1 - p)$ ... (1)

i.e., $p = p + x\frac{dp}{dx} + a\frac{dp}{dx} - 2ap\frac{dp}{dx}$, $\left[p = \frac{dy}{dx}\right]$ i.e., $(x + a - 2ap)\frac{dp}{dx} = 0$ ... (2)

From (2) we have either $\frac{dp}{dx} = 0$ or $x + a - 2ap = 0$

If $\frac{dp}{dx} = 0$, then $dp = 0$ and integrating $p = c$, c being an arbitrary constant.

Substituting $p = c$ in (1), the general solution is given by $y = cp + ac(1 - c)$, c being an arbitrary constant.

Again, $x + a - 2ap = 0$ gives $p = \frac{x+a}{2a}$. Substituting $p = \frac{x+a}{2a}$ in (1), we have the required singular solution as $y = \frac{x+a}{2a}x + a\frac{x+a}{2a}\left(1 - \frac{x+a}{2a}\right)$

i.e., $4ay = 2x(x + a) + (x + a)(a - x) = (x + a)(2x + a - x)$ i.e., $(x + a)^2 = 4ay$.

Which shows that the family of straight lines determined by the general solution is a tangent of the corresponding singular solution.

4. Substituting $y = u$ and $xy = v$, reduce the equation $x^2p^2 + y(2x + y)p + y^2 = 0$ into Clairaut's equation. Hence find its general and singular solutions.
[CP 2017]

Solution Here the given differential equation is $x^2p^2 + y(2x + y)p + y^2 = 0$ ... (1)

We are to substitute $y = u$ and $xy = v$ i.e., $x = \frac{v}{u}$ and $y = u$.

Then $\frac{dy}{dx} = \frac{du}{dx}$ and $y + x\frac{dy}{dx} = \frac{dv}{dx}$

Now, $\frac{dv}{du} = \frac{dv}{dx}\Big/\frac{du}{dx} = \frac{y + x\frac{dy}{dx}}{\frac{dy}{dx}} = \frac{u + \frac{v}{u}p}{p}$, since $p = \frac{dy}{dx}$

i.e., $q = \frac{u}{p} + \frac{v}{u}$, where $q = \frac{dv}{du}$ i.e., $p = \frac{u}{q - \frac{v}{u}} = \frac{u^2}{qu - v}$

with these substitutions, (1) become $\frac{v^2}{u^2}\cdot\frac{u^4}{(qu - v)^2} + u\left(\frac{2v}{u} + u\right)\frac{u^2}{qu - v} + u^2 = 0$

i.e., $v^2u^2 + (2v + u^2)u^2(qu - v) + u^2(qu - v)^2 = 0$

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i.e., $v^2 + (2v + u^2) + (qu - v)(qu - v) = 0$ [assuming $u \neq 0$]

i.e., $v^2 + (qu - v)(u^2 + qu + v) = 0$ i.e., $v^2 + u^2(qu - v) + q^2u^2 - v^2 = 0$

i.e., $u^2(q^2 + uq - v) = 0$ i.e., $v = uq + q^2$ [assuming $u \neq 0$] ... (2)

Equation (2) is the required differential equation in Clairaut's form with $q = \frac{dv}{du}$

Differentiating (2) with resecpt to u, we get

$$q = q + u\frac{dq}{du} + 2q\frac{dq}{du} \quad i.e., \left(q + \frac{u}{2}\right)\frac{dq}{du} = 0 \qquad ... (3)$$

From (3) we have either $\frac{dq}{du} = 0$ or $q + \frac{u}{2} = 0$

Now $\frac{dq}{du} = 0$ gives $q = c$, an arbitrary constant.

Replacing $q = c$ in (2) we have, $v = uc + c^2$

i.e., $xy = cy + c^2$, which is the required general solutions.

Again when $q + \frac{u}{2} = 0$, then $q = -\frac{u}{2}$.

Substituting, $q = \frac{u}{2}$ in (2), we have

$$v = u\left(-\frac{u}{2}\right) + \frac{u^2}{4} = -\frac{u^2}{4} \quad i.e., u^2 + 4v = 0 \ i.e., y(y + 4x) = 0$$

Hence the singular solution of the given equation is either $y = 0$ or $y + 4x = 0$

The general solution of the given equation is $xy = cy + c^2 \Rightarrow c^2 + cy - xy = 0$ which is a quadratic equation of c.

This discriminant of the equation must vanish.

So, $y^2 + 4xy = 0$ or $y(y + 4x) = 0$

Therefore the singular solution is either $y + 4x = 0$ or $y = 0$ [which is also a particular solution]

5. **Obtain general and singular solution of $py = p^2(x - b) + a$.** [CP 2006, 2011, 2014]

Solution Here the given differential equation $py = p^2(x - b) + a$

$$i.e., y = px - bp + \frac{a}{p} \qquad ... (1)$$

Clearly differential equation (1) is Clairaut's form.

Differentiating (1) with respect to x, we have

$$p = p + x\frac{dp}{dx} - b\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}; p \equiv \frac{dy}{dx} \quad i.e., \left(x - b - \frac{a}{p^2}\right)\frac{dp}{dx} = 0 \qquad ... (2)$$

From (2) we have either $\frac{dp}{dx} = 0$ or, $x - b - \frac{a}{p^2} = 0$

when $\frac{dp}{dx} = 0$, then $dp = 0$ and integrating $p = c$, c being a constant

So substituting $p = c$ in (1), the required general solution is

$y = cx - bc + \frac{a}{c}$, c being an arbitrary constant and a, b are two given constants.

Next to find singular solution if, $x - b - \frac{a}{p^2} = 0$, then $x - b = \frac{a}{p^2}$ and $p^2 = \frac{a}{x - b}$

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Then from (1) we have, $y = p(x - b) + \frac{a}{p} = p \cdot \frac{a}{p^2} + \frac{a}{p}$ $\left[\text{since } x - b = \frac{a}{p^2}\right]$

i.e., $y^2 = \left(\frac{2a}{p}\right)^2 = \frac{4a^2}{a}(x - b)$ i.e., $y^2 = 4a(x - b)$,

Which is the required singular solution for a, b are two given constants.

6. **Obtain general and singular solution for $y = px + p^n$** [CP 2015]

Solution Here the given differential equation is $y = px + p^n$ ... (1)

Clearly the differential equation (1) is in Clairaut's form

To solve (1), differentiating (1) w.r. to x, we have

$$p = p + x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{i.e., } \left(x + np^{n-1}\right)\frac{dp}{dx} = 0 \qquad \text{... (2)}$$

From (2) we have, either $\frac{dp}{dx} = 0$ or $x + np^{n-1} = 0$

when $\frac{dp}{dx} = 0$ then $dp = 0$ and on integrating $p = c$, an arbitrary constant.

Therefore, the required general solution is $y = cp + c^n$, c being an arbitrary constant

To find the singular solution, we have, $x + np^{n-1} = 0$ i.e., $p^{n-1} = -\frac{x}{n}$

Substituting this result in (1), we have

$$y = px + p\left(-\frac{x}{n}\right) = \frac{(n-1)}{n}px \quad \text{i.e., } y^{n-1} = \left(\frac{n-1}{n}\right)^{n-1} p^{n-1}x^{n-1} = \left(\frac{n-1}{n}\right)^{n-1}\left(-\frac{x}{n}\right)x^{n-1}$$

$$\text{i.e., } \left(\frac{y}{n-1}\right)^{n-1} = -\frac{x^n}{n^x} \quad \text{i.e., } \left(\frac{x}{n}\right)^n + \left(\frac{y}{n-1}\right)^{n-1} = 0$$

Which is the required singular solution.

7. **Obtain the general and singular solution of $yp = xp^2 - p - 2$** [CP 2015]

Solution Here the given equation is $yp = xp^2 - p - 2$ i.e., $y = px - 1 - \frac{2}{p}$ ... (1)

Clearly the differential equation (1) is Clairaut's form.

To solve (1), differentiating (1) w.r. to x, we have

$$p = p + x\frac{dp}{dx} + \frac{2}{p^2}\frac{dp}{dx} \quad \text{i.e., } \left(x + \frac{2}{p^2}\right)\frac{dp}{dx} = 0 \qquad \text{... (2)}$$

From (2) we have either $\frac{dp}{dx} = 0$ or $x + \frac{2}{p^2} = 0$

Now, $\frac{dp}{dx} = 0$ gives $dp = 0$ and on integration $p = c$, c being an arbitrary constant

Substituting $p = c$ in (1), the required general solution becomes

$$y = cx - 1 - \frac{2}{c} \quad \text{i.e., } cy = c^2x - c - 2, \; c \text{ being an arbitrary constant}$$

To find the singular solution, we have, $x + \frac{2}{p^2} = 0$ i.e., $\frac{2}{p} = -px$,

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Therefore from (1), $y = px - 1 + px$ *i.e.*, $y + 1 = 2px$

i.e., $(y+1)^2 = 4x^2p^2 = 4x^2\left(-\frac{2}{x}\right)$ $\left[\text{since } x + \frac{2}{p^2} = 0 \text{ then } p^2 = -\frac{2}{x}\right]$

i.e., $(y+1)^2 + 8x = 0$, which is the required singular solution.

8. **Find the general and singular solution of the differential equation $\sin px \cos y = \cos px \sin y + p$.** [CP 201..]

Solution Here, the given differential equation is $\sin px \cos y = \cos px \sin y + p$

i.e., $p = \sin px \cos y - \cos px \sin y = \sin(px - y)$

i.e., $px - y = \sin^{-1}p$ *i.e.*, $y = px - \sin^{-1}p$ ... (1)

clearly the equation (1) is in Clairaut's form.

To solve (1), differentiating (1) with respect to x,

we have, $p = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$, where $p = \frac{dy}{dx}$

i.e., $\left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$ ... (2)

From (2) we have either $\frac{dp}{dx} = 0$ or, $x - \frac{1}{\sqrt{1-p^2}} = 0$

when $\frac{dp}{dx} = 0$, then $dp = 0$ and on integrating, we have $p = c$, where c is an arbitrary constant.

Substituting $p = c$ in (1), required general solution is

$y = cx - \sin^{-1}c$, c being an arbitrary constant.

To obtain singular solution we have, $x - \frac{1}{\sqrt{1-p^2}} = 0$ *i.e.*, $\frac{1}{1-p^2} = x^2$

i.e., $p = \frac{\sqrt{x^2-1}}{x}$ [considering +ve sign only] ... (3)

Eliminating p between (1) and (3), the required singular solution is

$y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}$.

9. **Obtain the general and singular solution of $(y - px)(p - 1) = p$** [CP 2013]

Solution Here the given differential equation is $(y - px)(p - 1) = p$

i.e., $py - y - p^2x + px = p$ *i.e.*, $y = px + py - p^2x - p$ ... (1)

Clearly the equation (1) is in Clairaut's form. Differentiating (1) with respect to x, we have

$p = p + x\frac{dp}{dx} + p^2 + y\frac{dp}{dx} - p^2 - 2px\frac{dp}{dx} - \frac{dp}{dx}$, $\left[p = \frac{dy}{dx}\right]$

i.e., $(x + y - 1 - 2px)\frac{dp}{dx} = 0$ ... (2)

Thus we have from (2) either $\frac{dp}{dx} = 0$ or $x + y - 1 - 2px = 0$

To have the general solution, we have $\frac{dp}{dx} = 0$, giving $dp = 0$ and on integrating we have $p = c$, c being an arbitrary constant.

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Substituting $p = c$ in (1), the required general solution thus becomes

$y = cx + cy - c^2x - c$ *i.e.*, $(y - cx)(c - 1) = c$, c being an arbitrary constant.

To find the singular solution, we have, $x + y - 1 - 2px = 0$ *i.e.*, $p = \frac{x+y-1}{2x}$

Eliminating p between (1) and (3), the required singular solution is as follows :

$$\left(y - \frac{x+y-1}{2}\right)\left(\frac{x+y-1}{2x} - 1\right) = \frac{x+y-1}{2x} \quad i.e., (y - x + 1)(y - x - 1) = 2(x + y - 1)$$

i.e., $(x - y)^2 - 1 = 2(x + y) - 2$ *i.e.*, $(x - y)^2 + 1 = 2(x + y)$.

10. **Obtain the general and singular solution of $y = px + \frac{m}{p}$**

[CP 1995, 1997, 2009, 2012, BP 1994]

Solution Here the given differential equation is $y = px + \frac{m}{p}$ ... (1)

Clearly the equation (1) is in Clairaut's form. Therefore, to solve (1) differentiating (1) with respect to x, we have $p = p + x\frac{dp}{dx} - \frac{m}{p^2}\frac{dp}{dx}$ *i.e.*, $\left(x - \frac{m}{p^2}\right)\frac{dp}{dx} = 0$ ... (2)

Therefore, from (2) we have, either $\frac{dp}{dx} = 0$ or $x - \frac{m}{p^2} = 0$

When $\frac{dp}{dx} = 0$ then $dp = 0$ and on integrating $p = c$, c being an arbitrary constant.

To find the general solution, substituting $p = c$ in (1), we get $y = cx + \frac{m}{c}$ which is the required general solution.

Again to find singular solution, we have $x - \frac{m}{p^2} = 0$ *i.e.*, $\frac{m}{p} = px$ and $p^2 = \frac{m}{x}$,

Thus from (1), we have $y = px + \frac{m}{p} = px + px = 2px$ *i.e.*, $y^2 = 4p^2x = 4x^2\left(\frac{m}{x}\right)$

i.e., $y^2 = 4mx$, which is the required singular solution.

11. **Solve completely $p = \log(px - y)$**

Solution Here the given differential equation is $p = \log(px - y)$

i.e., $px - y = e^p$ *i.e.*, $y = px - e^p$ ... (1)

Clearly, the equation (1) is in Clairaut's form.

Differentiating (1) with respect to x, we have $p = p + x\frac{dp}{dx} - e^p\frac{dp}{dx}$ *i.e.*, $(x - e^p)\frac{dp}{dx} = 0$ ... (2)

From (2), we have either $\frac{dp}{dx} = 0$ or $x - e^p = 0$

When $\frac{dp}{dx} = 0$, then $dp = 0$ and on integrating $p = c$, c being an arbitrary constant.

Putting $p = c$ in (1), the required general solution becomes as $y = cx - e^c$

For singular solution, we have $x - e^p = 0$ *i.e.*, $p = \log x$ ... (3)

Eliminating p between (1) and (3), required singular solution becomes

$y = x\log x - x$ *i.e.*, $y = (\log x - 1)x$.

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12. **Obtain the general and singular solution of $y = px + \sqrt{a^2p^2 - b^2}$**

Solution Here the given differential equation is $y = px + \sqrt{a^2p^2 - b^2}$ ... (1)

Differentiating (1) with respect to x, we have $\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{2a^2p}{2\sqrt{a^2p^2 - b^2}}\frac{dp}{dx}$

i.e., $p = p + \left(x + \frac{a^2p}{\sqrt{a^2p^2 - b^2}}\right)\frac{dp}{dx}$ i.e., $\left(x + \frac{a^2p}{\sqrt{a^2p^2 - b^2}}\right)\frac{dp}{dx} = 0$

Now either $\frac{dp}{dx} = 0$ i.e., $dp = 0$ on integrating $p = c$ (c is an arbitrary constant) ... (2)

∴ The general solution is $y = cx + \sqrt{a^2c^2 - b^2}$, c being an arbitrary constant.

Now, $x + \frac{a^2p}{\sqrt{a^2p^2 - b^2}} = 0$ i.e., $\frac{x}{a} = -\frac{ap}{\sqrt{a^2p^2 - b^2}}$ ... (3)

Now, from (1), we have $y = -\frac{a^2p^2}{\sqrt{a^2p^2 - b^2}} + \sqrt{a^2p^2 - b^2}$ i.e., $y = \frac{-b^2}{\sqrt{a^2p^2 - b^2}}$

∴ $\frac{y}{b} = -\frac{b}{\sqrt{a^2p^2 - b^2}}$ ... (4)

Now squaring (3) and (4) and subtracting, we get, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2p^2 - b^2}{a^2p^2 - b^2} = 1$

Therefore the singular solution is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which is the equation of hyperbola.

13. **Reduce the equation $(p - 1)e^{3x} + p^3e^{2y} = 0$ into the Clairaut's form by using the substitution $e^x = u$ and $e^y = v$. Hence find the general solution.**

Solution Here the given differential equation is $(p - 1)e^{3x} + p^3e^{2y} = 0$ ... (1)

Put $e^x = u$ and $e^y = v$, then $du = e^x dx$ and $dv = e^y dy$

So, $\frac{dv}{du} = \frac{e^y}{e^x}\cdot\frac{dy}{dx} = \frac{v}{u}\cdot p$ i.e., $q = \frac{v}{u}\cdot p$ when $\frac{dv}{du} = q$ i.e., $p = \frac{qu}{v}$ ... (2)

From (1) and (2), $\left(\frac{qu}{v} - 1\right)u^3 + \frac{q^3u^3}{v^3}v^2 = 0$ i.e., $\frac{qu}{v} - 1 + \frac{q^3}{v} = 0$ i.e., $v = qu + q^3$ ... (3)

which is the required Clairaut's equation

Now differentiating (3) with respect to u, we get $\frac{dv}{du} = q + u\frac{dq}{du} + 3q^2\frac{dq}{du}$

i.e., $q = q + (u + 3q^2)\frac{dq}{du}$ $\left[\text{since } q = \frac{dv}{du}\right]$ i.e., $(u + 3q^2)\frac{dq}{du} = 0$

Now, $\frac{dq}{du} = 0$ i.e., $dq = 0$ on integration $q = c$ (c is an arbitrary constant)

From (3) $v = cu + c^3$ i.e., $e^y = ce^x + c^3$, which is the required general solution.

Note

To find the singular solution, $u + 3q^2 = 0$ i.e., $q^2 = -\frac{u}{3}$ ... (4)

Eliminating q between (3) and (4), we get the singular solution.

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Exercise

- **Obtain the general and singular solution of (1-7) :**

1. $y = px + \sqrt{1+p^2}$ [CH 1987]
2. $y = px + \sqrt{a^2p^2 + b^2}$ [CP 2010]
3. $y = px + \frac{a}{p}$ [CP 2009]
4. $(y + 1)p - xp^2 + 2 = 0$
5. $y = (x - a)p^2 + (x - y)p$
6. $y = px + e^p$
7. $1 - p^2 = \sin^{-1}(px - y)$
8. Reduce the equation $(p + 1)^2 = (px + y)(px^2 + y^2)$ into Clairaut's form by using the substitutions $u = xy$ and $v = x + y$. Hence find its solutions. [CH 1992]
9. Find the general and singular solution of $1 - p^2 = \sin^{-1}(px - y)$. [CH 1989]
10. Reduce the equation $(px^2 + y^2)(px + y) = (p + 1)^2$ to Clairaut's form by substitutions $u = xy$ and $v = x + y$. Hence find its complete solution. [CP 1992]
11. Using the transformations $x^2 = u, y^2 = v$, solve the equation $(px - y)(x - py) = 2p$ completely. [CP 1995, 2015]
12. Solve : $x + py = ap^2$

    [*Hint :* Here $y = -\frac{x}{p} + ap$

    So differentiating w.r.t. to x, we have

    $$p = -\frac{1}{p} + \frac{x}{p^2}\frac{dp}{dx} + a\frac{dp}{dx}$$

    i.e. $\frac{dp}{dx} - \frac{x}{p(p^2+1)} = \frac{ap}{1+p^2}$

    The equation is linear in x and so its general solution is $\frac{x\sqrt{1+p^2}}{p} = c + a\log(p + \sqrt{1+p^2})$, c being an arbitrary constant. Therefore the general solution can be obtained by eliminating p from given equation $x + py = ap^2$ and

    $$\frac{x\sqrt{1+p^2}}{p} = c + a\log(p + \sqrt{1+p^2})]$$

Answers

1. $x^2 + y^2 = 1,\ y = cx + \sqrt{1+c^2}$
2. $y = cp + \sqrt{a^2c^2 + b^2}$ [general solution]

   $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [singular solution]
3. $y = cx + \frac{a}{c}$ [general solution]

   $y^2 = 4ax$ [singular solution]
4. $(y + 1)c - xc^2 + 2 = 0$ [general solution]

   $(y + 1)^2 + 8x = 0$ [singular solution]
5. $y = (x - a)c^2 + (x - y)c$

   $(x + y)^2 = 4ay$
6. $y = cx + e^c$

   $y = x(\log x - 1)$
7. $y = cx - \sin(1 - c^2)$

   $y = px - \sin(1 - p^2)$,

   $x + 2p\cos(1 - p^2) = 0$
8. $cxy - c^2(x + y) + 1 = 0$
9. $y = cx - \sin(1 - c^2)$ [general solution]

   singular solution is eliminating p from $y = px - \sin(1 - p^2), x + 2p\cos(1 - p^2) = 0$
10. $cxy - c^2(x + y) + 1 = 0$
11. $(c^2x^2 + y^2) + c(2 - x^2 - y^2) = 0$ is the general solution and $(2 - x^2 - y^2)^2 = 4x^2y^2$ is the singular solution.
12. Eliminating p from $x + py = ap^2$,

    $$\frac{x\sqrt{1+p^2}}{p} = c + a\log(p + \sqrt{1+p^2})$$

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7 CHAPTER

Higher Order Differential Equation with Constant Coefficients

7.1 Higher Order Differential Equations

In Fumdamental Mathematics (SEM-1), we have already provided concept of ordinary differential equation, both of first and second order. Recalling from there, if x is the independent variable and $y \equiv y(x)$ be the dependent variable, then the n-th order differential equation will be the following form :

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + P_{n-1}(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_1(x)\frac{dy}{dx} + P_0(x)y = Q(x) \quad \ldots (1)$$

The differential equation (1) is said to be a linear differential equation of order n provided $P_n(x) \neq 0$ and all of $P_i(x)$, $i = 0, 1, \ldots, n$ and $Q(x)$ are functions of x only. If all the $P_i(x)$, $i = 0, 1, 2, \ldots, n$ are of constant values, then the differential equation (1) is said to be n-order linear differential equation with constant coefficient. **Therefore a simplified form of n order linear differential equation with constant co-efficient is as follows :**

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_1\frac{dy}{dx} + a_0 y = Q(x) \quad \ldots (2)$$

where $a_0, a_1, \ldots, a_{n-1}$ are n constants and $Q(x)$ is a function of x.

Thus in particular for $n = 2, 3$, and 4, we have an ordinary differential equation with constant coefficient whose order is 2, 3 and 4 respectively. Hence the general form of a second order differential equation with constant coefficient can be written as follows :

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = Q(x) \quad \ldots (3)$$

The solution of the differential equation (3) has two parts, one is known as the complementary function (C.F.) due to the differential equation of the reduced form $\left(i.e.\ \frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 = 0\right)$ and the second due to the non-zero function $\phi(x)$ present in the right hand side which is known as the particular integral (P.I.). We will denote the C.F. by y_C and the P.I. by y_P. The solution of the equation $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$ $\ldots (4)$

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contains two arbitrary constants, since the order of the differential equation (4) is two. Further the P.I., y_P is arising due to the non-zero right hand side function $\phi(x)$, and the P.I. would not contain any arbitrary constant. Therefore y_P should satisfy the equation (3) and we have

$$\frac{d^2y_P}{dx^2} + a_1\frac{dy_P}{dx} + a_2y_P = Q(x) \quad \text{... (5)}$$

Similarly, we have $\frac{d^2y_C}{dx^2} + a_1\frac{dy_C}{dx} + a_2y_C = 0$ ... (6)

Theorem : The general solution of the differential equation (2) is $y = y_C + y_P$

We have, $\frac{d^2}{dx^2}(y_C + y_P) + a_1\frac{d}{dx}(y_C + y_P) + a_2(y_C + y_P)$

$$= \frac{d^2y_C}{dx^2} + \frac{d^2y_P}{dx^2} + a_1\frac{dy_C}{dx} + a_1\frac{dy_P}{dx} + a_2y_C + a_2y_P$$

$$= \left(\frac{d^2y_C}{dx^2} + a_1\frac{dy_C}{dx} + a_2y_C\right) + \left(\frac{d^2y_P}{dx^2} + a_1\frac{dy_P}{dx} + a_2y_P\right)$$

$= 0 + Q(x)$ [from (5) and (6)]

$= Q(x)$

Further the solution $y = y_C + y_P$ contains two arbitrary constants and since equation (3) is a second order differential equation which is satisfied by $y = y_C + y_P$, therefore we may conclude that $y = y_C + y_P$ is the general solution or complete primitive of the differential equation (3).

7.2 Solution Procedure of Second Order Differential Equation with Constant Coefficient

First we reduced the differential equation (3) by substituting, $Q(x) = 0$, then we have

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

The general solution of (4) will be the C.F. of the original equation (3). Below we describe the process to find the general solution of the differential equation (4).

Let $y = ce^{mx}$ ($c \neq 0$) be a trial solution of the differential equation (4).

Then, $\frac{dy}{dx} = cme^{mx}$, $\frac{d^2y}{dx^2} = cm^2e^{mx}$ and substituting those in (4) we have

$$c(m^2 + a_1m + a_2)e^{mx} = 0$$

i.e., $m^2 + a_1m + a_2 = 0$ [since $ce^{mx} \neq 0$] ... (7)

Equation (7) is known as the auxiliary equation (A.E.)

Equation (7) is a quadratic equation on m.

Therefore, from algebra we know that, it has two roots.

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Let $m = m_1, m_2$ be two roots of (7). Then we have following three possibilities :

Case-1 : m_1, m_2 are real and unequal.

Case-2 : m_1, m_2 are real and equal so that $m = m_1 = m_2$

Case-3. m_1, m_2 are complex conjugate and let $m_1 = a + ib$, $m_2 = a - ib$, where a, b are two real numbers.

Let us discuss the above cases one by one :

- **Case-1 :** If m_1, m_2 are real and unequal, then the general solution of the differential equation (4) are given by $y(x) = A_1e^{m_1x} + A_2e^{m_2x}$, where A_1, A_2 are two arbitrary constants. ... (8)
- **Case-2 :** If m_1, m_2 are real and equal, so that $m = m_1 = m_2$, then the general solution of the differential equation (4) is given by $y(x) = (A_1 + A_2x)e^{mx}$, where A_1, A_2 are two arbitrary constants. ... (9)
- **Case-3 :** If m_1, m_2 are complex conjugate *i.e.* $m_1 = a + ib$, $m_2 = a - ib$, then the general solution of the differential equation (4) is given by $y(x) = e^{ax}(A_1\cos bx + A_2\sin bx)$... (10) where A_1, A_2 are two arbitrary constants.

Sometimes the above form of solution can be taken as $y(x) = Ae^{ax}\cos(bx + B)$ or $y(x) = Ae^{ax}\sin(bx + B)$, where A and B are two arbitrary constants.

Hence depending on nature of the roots of the auxiliary equation, solution of the differential equation (4) is obtained.

Illustration-1 : Find the complementary function of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = x^2$. [CP 2009]

Solution : Here the given differential equation is $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = x^2$ ... (11)

The complementary function of (11) will be the general solution of the reduced equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 \quad \text{... (12)}$$

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (12)

Then the A.E. of (12) becomes as $m^2 - 6m + 9 = 0$ *i.e.* $(m - 3)^2 = 0$ giving $m = 3, 3$.

Since the two roots of the A.E. are real and equal, therefore the C.F. of given differential equation is $y = (A + Bx)e^{3x}$, where A, B are two arbitrary constants.

7.3 Complementary Functions of Higher Order Differential Eqution

However if the order of the differential equation is more than two (three, four, five and so on) than the determination of complementary function would also be same procedure as described earlier. For example, if the differential equation is of third order having constant coefficients than the auxiliary equation would be an algebraic equation of degree three. Further the roots of that auxiliary equation would give the required complementary function using the rules stated in above section. Similarly, if an ordinary differential equation with constant coefficient is of degree four and higher order, then also by similar manner complementary function can be obtained. We illustrate this in following exam

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Illustration-2 : Solve : $\frac{d^4y}{dx^4} - a^2y = 0$ [CH 1994]

Solution : Here the given differential equation is $\frac{d^4y}{dx^4} - a^4y = 0$ ... (13)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (13)

Then the auxiliary equation of (13) is

$$m^4 - a^4 = 0 \text{ i.e., } (m^2 - a^2)(m^2 + a^2) = 0 \text{ giving } m = \pm a, \pm ia.$$

So, the required general solution (which is equivalent to the complementary function) of given differential equation is :

$y = Ae^{ax} + Be^{-ax} + C\cos ax + D\sin ax$, where A, B, C, D are four arbitrary constants.

7.4 Particular Integral

We consider a second order linear differential equation with constant coefficient as follows:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = Q(x) \qquad \text{... (14)}$$

where a_1, a_2 are two constants and $Q(x)$ is a function of x only.

Introducing the differential operator $D \equiv \frac{d}{dx}$, the above differential equation can be rewritten as : $(D^2 + a_1D + a_2)y = Q(x)$ ... (15)

i.e., $f(D)y = Q(x)$ ... (16)

where $f(D)$ is a quadratic polynomial of the operator D such that $f(D) = D^2 + a_1D + a_2$.

Therefore $f(D)$ is also an operator. The P.I. will be determined by solving $\frac{1}{f(D)}Q(x)$

The general solution of (10) having two parts, one due to the corresponding homogeneous equation, which is commonly known as the complementary function (C.F.) and second due to the non-zero right hand side function $\phi(x)$ which is commonly known as particular integral (P.I.). Thus C.F. of the given equation will be the general solution of the corresponding reduced equation of (14), i.e. in other words C.F. will be general solution of

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2y = 0.$$

Solution procedure of above differential equation is discussed in earlier section. Proceeding same way, let $y = y_C$ (which contain two arbitrary constants) be the C.F.

Here we mainly interested to find the P.I. of a non-homogeneous equation. The process to find P.I. for various types of Q(x) will be discussed in next section. Let $y = y_P$ (which contains no arbitrary constant) be the P.I.

Therefore the general solution will be $y = y_C + y_P$ which will contain two arbitrary constants.

7.5 Dealing with the Operator D

The operator D stands for $\frac{d}{dx}$ and D^2 stands for $\frac{d^2}{dx^2}$ and so on. Therefore the operator DV stands for one time derivative of the function V with respect to x and the operator D^2V

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stands for twice derivative of the function V with respect to x.

Thus $D(x) = 1$, $D(2x^2) = 4x$, $D^2(x^3) = 6x$, $D^2(\sin x) = -\sin x$, $D^2(e^{ax}) = a^2e^{ax}$ and so on.

Similarly, $(D + a)(x) = D(x) + ax = 1 + ax$

$$(D + a)e^{ax} = D(e^{ax}) + ae^{ax} = ae^{ax} + ae^{ax} = 2ae^{ax}$$

$$(D^2 + 1)e^x = D^2(e^x) + e^x = e^x + e^x = 2e^x$$

$$(D^2 + a^2)\sin ax = D^2(\sin ax) + a^2\sin ax = -a^2\sin ax + a^2\sin ax = 0 \text{ and so on.}$$

Thus from (11) and (12), $f(D)\phi(x) = (D^2 + p_1D + p_2)\phi(x) = \phi''(x) + p_1\phi'(x) + p_2\phi(x)$

Illustration-3 : Find $f(D)\phi(x)$, where $f(D) = D^2 + p_1D + p_2$ and (i) $\phi(x) = ax^2 + bx + c$ (ii) $e^{ax} + \sin bx$ (iii) $\cos(ax + b)$

Solution : (i) We have, $\phi(x) = ax^2 + bx + c$

Therefore $f(D)\phi(x) = (D^2 + p_1D + p_2)(ax^2 + bx + c)$

$$= D^2(ax^2 + bx + c) + p_1D(ax^2 + bx + c) + p_2(ax^2 + bx + c)$$

$$= 2a + p_1(2ax + b) + p_2(ax^2 + bx + c) = ap_2x^2 + (2ap_1 + bp_2)x + 2a + bp_1 + cp_2$$

(ii) We have, $\phi(x) = e^{ax} + \sin bx$

Therefore $f(D)\phi(x) = (D^2 + p_1D + p_2)(e^{ax} + \sin bx)$

$$= (D^2 + p_1D + p_2)e^{ax} + (D^2 + p_1D + p_2)(\sin bx)$$

$$= a^2e^{ax} + p_1ae^{ax} + p_2e^{ax} + (-b^2\sin bx) + p_1b\cos bx + p_2\sin bx$$

$$= (a^2 + ap_1 + p_2)e^{ax} + (p_2 - b^2)\sin bx + bp_1\cos bx$$

(iii) We have, $\phi(x) = \cos(ax + b)$

Therefore $f(D)\phi(x) = (D^2 + p_1D + p_2)\cos(ax + b)$

$$= -a^2\cos(ax + b) - p_1a\sin(ax + b) + p_2\cos(ax + b)$$

$$= (p_2 - a^2)\cos(ax + b) - ap_1\sin(ax + b)$$

Since D stands for the differential operator $\frac{d}{dx}$ therefore $\frac{1}{D}$ stands for integration operator.

Therefore $\frac{1}{D}(x) = \frac{x^2}{2}$, $\frac{1}{D}(e^{ax}) = \frac{e^{ax}}{a}$ and so on.

Similarly, $\frac{1}{D^2}(x) = \frac{1}{D}\left(\frac{1}{D}(x)\right) = \frac{1}{D}\left(\frac{x^2}{2}\right) = \frac{x^3}{6}$.

$$\frac{1}{D^2}(\sin ax) = -\frac{\sin ax}{a^2}, \quad \frac{1}{D^2}(e^{bx}) = \frac{e^{bx}}{b^2} \text{ and so on.}$$

In a word $\frac{1}{D}\phi(x)$ gives the solution of the equation $Dy = \phi(x)$ and $\frac{1}{D^2}\phi(x)$ gives the solution of $D^2y = \phi(x)$.

Hence, $\frac{1}{f(D)}\phi(x)$ will give the solution of the differential equation $f(D)y = \phi(x)$.

Thus P.I. of the differential equation (1) will be determined by solving $\frac{1}{f(D)}\phi(x)$.

In next section we will describe some rules to find the values of $\frac{1}{f(D)}\phi(x)$.

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7.6 Rules for finding P.I.

The solution obtained for the equation $\frac{1}{f(D)}\phi(x)$ without any arbitrary constants, known as the particular integral or P.I. of the differential equation $f(D)y = \phi(x)$ where $f(D) = D^2 + p_1 D + p_2$ and $\phi(x)$ is a function of x.

- **Rule-1 :** If $\phi(x) = e^{ax}$, then $\frac{1}{f(D)}\phi(x) = \frac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$

- **Proof :** Since $D(e^{ax}) = ae^{ax}$ and $D^2e^{ax} = a^2e^{ax}$, then $f(D)e^{ax} = f(a)e^{ax}$.

  Therefore, $f(D)\left\{\frac{e^{ax}}{f(a)}\right\} = e^{ax}$, provided $f(a) \neq 0$ *i.e.*, $\frac{1}{f(D)}(e^{ax}) = \frac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$.

- **Illustration-4 : Find P.I. of $f(D)y = e^{2x}$ where $f(D) = D^2 + D + 1$** [CP 1986]

- **Solution :** We have, $y_P = \frac{1}{f(D)}e^{2x} = \frac{1}{D^2+D+1}e^{2x} = \frac{1}{4+2+1}(e^{2x}) = \frac{1}{7}e^{2x}$.

  Hence the required P.I. is $\frac{1}{7}e^{2x}$.

- **Rule-2 :** If $\phi(x) = e^{ax}\,\Psi(x)$, then $\frac{1}{f(D)}\phi(x) = e^{ax}\frac{1}{f(D+a)}\psi(x)$.

- **Rule-3 :** If $\phi(x)$ is polynomial function of x, then $\frac{1}{f(D)}\phi(x)$ can be obtained by writing $\frac{1}{f(D)}$ as $[f(D)]^{-1}$ and then expanding it ascending powers of the differential operator D.

- **Rule-4 :** If $\phi(x) = \sin(ax + b)$ or $\phi(x) = \cos(ax + b)$ and if the equivalent form of $f(D)$ is $g(D^2)$, then $\frac{1}{f(D)}\sin(ax+b) = \frac{1}{g(-a^2)}\sin(ax+b)$

  and $\frac{1}{f(D)}\cos(ax+b) = \frac{1}{g(-a^2)}\cos(ax+b)$, provided $g(-a^2) \neq 0$.

Note

$\frac{1}{f(D)}\cos(ax) = \frac{1}{g(D^2)}\cos(ax) = \frac{1}{g(-a^2)}\cos(ax)$ and

$\frac{1}{f(D)}\sin(ax) = \frac{1}{g(D^2)}\sin(ax) = \frac{1}{g(-a^2)}\sin(ax)$ provided $g(-a^2) \neq 0$.

- **Illustration-5 : Find the particular integral of $(D^2+4D+4)y = x^3e^{-2x}$** [CP 2014]

- **Solution :** The P.I. of given equation is $y_P = \frac{1}{(D^2+4D+4)}(x^3e^{-2x})$

$$= \frac{1}{(D+2)^2}(e^{-2x}x^3) = e^{-2x}\frac{1}{((D-2)+2)^2}x^3$$

$$= e^{-2x}\frac{1}{D^2}x^3 = e^{-2x}\frac{1}{D}\left(\frac{x^4}{4}\right) = \frac{x^5e^{-2x}}{20}$$

- **Illustration-6 : Find the P.I. of $(D^2 - 1)y = e^x$** [CP 2014]

- **Solution :** The P.I. of the given equation is $y_P = \frac{1}{D^2-1}e^x = \frac{1}{D^2-1}(e^x\cdot 1)$

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$$= e^x \frac{1}{(D+1)^2-1}(1) = e^x \frac{1}{D^2+2D}(1) = e^x \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1)$$

$$= e^x \frac{1}{2D}\left(1-\frac{D}{2}+\ldots\right)(1) \text{ [Expanding in infinite series]}$$

$$= e^x \frac{1}{2D}(1) = \frac{xe^x}{2}$$

■ **Alternative Method**

$$\text{P.I.} = \frac{1}{D^2-1}e^x = x\frac{1}{2D}e^x = x\cdot\frac{1}{2}\cdot e^x = \frac{xe^x}{2}$$

Illustration-7 : Find the P.I. of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$ [CP 2014]

Solution : If y_P be the P.I., then we have,

$$y_P = \frac{1}{D^2-2D}e^x \sin x = e^x \frac{1}{(D+1)^2-2(D+1)}(\sin x) = e^x \frac{1}{D^2-1}(\sin x)$$

$$= e^x \frac{1}{-1-1}(\sin x) = -\frac{e^x \sin x}{2}$$

Hence the P.I. is $-\frac{e^x \sin x}{2}$

Illustration-8 : Find the P.I. of $(D^2 - 6D + 9)y = 4x^2$

Solution : If y_P be the P.I., then we have, [CP 2014]

$$y_P = \frac{1}{D^2-6D+9}(4x^2) = 4\frac{1}{D^2-6D+9}(x^2) = \frac{4}{9}\left\{1+\frac{D^2-6D}{9}\right\}^{-1}(x^2)$$

$$= \frac{4}{9}\left\{1+\frac{D^2-6D}{9}-\left(\frac{D^2-6D}{9}\right)^2+\ldots\right\}(x^2) = \frac{4}{9}\left\{1+\frac{D^2-6D}{9}+\frac{36D^2}{81}-\ldots\right\}(x^2)$$

$$= \frac{4}{9}\left(x^2+\frac{2}{9}-\frac{12x}{9}+\frac{72}{81}\right) = \frac{4}{9}\left(x^2-\frac{4x}{3}+\frac{10}{9}\right) = \frac{4}{81}(9x^2-12x+10), \text{ which is required P.I.}$$

Rule-5 : If $\phi(x) = \sin ax\,\Psi(x)$ or $\phi(x) = \cos ax\,\Psi(x)$ then write $\sin ax$ and $\cos ax$ respectively as imaginary part of e^{iax} and real part of e^{iax}. Then proceed as Rule no. 2.

Illustration-9 : Find the P.I. of $\frac{d^2y}{dx^2} + y = \cos x$

Solution : If y_P be the P.I., then we have from the equation $y_P = \frac{1}{D^2+1}(\cos x)$

i.e., y_P = Real part of $\frac{1}{D^2+1}(e^{ix}.1)$ = Real part of $e^{ix}\frac{1}{(D+i)^2+1}(1)$

= Real part of $e^{ix}\frac{1}{D^2+2iD}(1)$ = Real part of $e^{ix}\frac{1}{2iD}\left(1+\frac{D}{2i}\right)^{-1}(1)$

= Real part of $\frac{e^{ix}}{2i}\frac{1}{D}\left(1-\frac{D}{2i}+\ldots\right)(1)$ = Real part of $\frac{e^{ix}}{2i}\frac{1}{D}(1)$

= Real part of $\frac{1}{2i}(\cos x + i\sin x)(x) = \frac{x\sin x}{2}$

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Similarly P.I. of $\frac{d^2y}{dx^2} + y = \sin x$ is imaginary part of $\frac{1}{D^2+1}(e^{ix})$ = Imaginary part of $\frac{1}{2i}(\cos x + i\sin x)(x) = -\frac{x\cos x}{2}$

Rule-6 : If $f(D) = (D - m)$, then $\frac{1}{D-m}\phi(x) = e^{mx}\int e^{-mx}\phi(x)dx$

Illustration-10 : Find the P.I. of $(D^2 + 1)y = \operatorname{cosec} x$ [BH 1998]

Solution : The P.I. of the given equation is

$$y_P = \frac{1}{D^2+1}\operatorname{cosec} x = \frac{1}{(D+i)(D-i)}\operatorname{cosec} x = \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\operatorname{cosec} x$$

i.e., $y_P = \frac{1}{2i}\left[\frac{1}{D-i}(\operatorname{cosec} x) - \frac{1}{D+i}(\operatorname{cosec} x)\right]$

Now, $\frac{1}{D-i}\operatorname{cosec} x = e^{ix}\int e^{-ix}\operatorname{cosec} x\,dx = e^{ix}\int\frac{\cos x - i\sin x}{\sin x}dx$

$$= e^{ix}\left[\int\cot x\,dx - i\int dx\right] = e^{ix}[\log\sin x - ix]$$

Also $\frac{1}{D+i}\operatorname{cosec} x = e^{-ix}\int e^{ix}\operatorname{cosec} x\,dx = e^{-ix}\int\frac{\cos x + i\sin x}{\sin x}dx$

$$= e^{-ix}\left[\int\cot x\,dx + i\int dx\right] = e^{-ix}[\log\sin x + ix]$$

Therefore, we have

$$y_p = \frac{1}{2i}[e^{ix}(\log\sin x - ix) - e^{-ix}(\log\sin x + ix)]$$

$$= \frac{1}{2i}(e^{ix} - e^{-ix})\log\sin x - \frac{x}{2}(e^{ix} - e^{-ix})$$

$= \sin x\log\sin x - x\cos x$, which is the required P.I.

Rule-7 : If $\phi(x) = xV$ where V is any function of x,

then $\frac{1}{f(D)}xV = \left\{x - \frac{1}{f(D)}\cdot f'(D)\right\}\frac{1}{f(D)}V$

Illustration-11 : If $\phi(x) = xe^{2x}$ and $f(D) = D^2 + 1$, then

$$\frac{1}{f(D)}xV = \frac{1}{D^2+1}xe^{2x} = \left\{x - \frac{1}{D^2+1}\cdot 2D\right\}\frac{1}{D^2+1}e^{2x}$$

$$= \left\{x - \frac{1}{D^2+1}\cdot 2D\right\} = \left\{x - \frac{1}{D^2+1}\cdot 2D\right\}\frac{e^{2x}}{2^2+1}$$

$$= \frac{1}{5}\left\{xe^{2x} - 2\frac{1}{D^2+1}D(e^{2x})\right\} = \frac{1}{5}\left\{xe^{2x} - 4\frac{1}{D^2+1}e^{2x}\right\}$$

$$= \frac{1}{5}\left\{xe^{2x} - 4\frac{e^{2x}}{2^2+1}\right\} = \frac{1}{5}\left\{xe^{2x} - \frac{4}{5}e^{2x}\right\}$$

Rule-8 : If $\phi(x) = x^nV$, then $\frac{1}{f(D)}x^nV = \left\{x - \frac{1}{f(D)}\cdot f'(D)\right\}^n\frac{1}{f(D)}V$, where n is a positive integer.

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Notes

(i) P.I. $= \frac{1}{f(D)} e^{ax}$ if $f(a) = 0$

$= x \cdot \frac{1}{f'(D)} e^{ax} = x \frac{e^{ax}}{f'(a)}$ provided $f'(a) \neq 0$

(ii) P.I. $= \frac{1}{f(D)} e^{ax}$ if $f(a) = f'(a) = 0$

$= x^2 \frac{1}{f''(D)} e^{ax} = x^2 \frac{e^{ax}}{f''(a)}$ provided $f''(a) \neq 0$

Illustrative Examples

1. **Find the complementary function of $9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = \cos 7x$** [CP 2008]

Solution Here the given differential equation is $9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = \cos 7x$ ... (1)

The reduced form of equation (1) is $9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of the differential equation (2).

Then the auxiliary equation (A.E.) of (2) becomes $9m^2 + 12m + 9 = 0$

i.e., $(3m + 2)^2 = 0$ giving $m = -\frac{2}{3}, -\frac{2}{3}$

Since the roots of A.E. are real and equal, therefore the general solution of the equation(2) [equivalent to the C.F. of equation (1)] is given by $y_c = (Ax + B)e^{-\frac{2}{3}x}$, where A and B are two arbitrary constants.

2. **Solve : $y'' + 4y' + 13y = 0$**

Solution Here the given equation is $y'' + 4y' + 13y = 0$ ... (1)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of the differential equation (1).

Then the auxiliary equation is $m^2 + 4m + 13 = 0$, giving $m = +\frac{1}{2}(-4 \pm \sqrt{16-52})$

i.e., $m = \frac{1}{2}(-4 \pm 6i) = -2 \pm 3i$.

Since the roots of the A.E. are complex conjugate, therefore the general solution of the given differential equation is

$y = e^{-2x}(A\cos 3x + B\sin 3x)$, where A, B are two arbitrary constants.

3. **Solve : $(D^3 - 1)y = 0$**

Solution Here the given differential equation is $(D^3 - 1)y = 0$ ... (1)

Let $y = c_1 e^{mx}$ $(c_1 \neq 0)$ be trial solution of (1)

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So, the auxiliary equation becomes as $m^3 - 1 = 0$ giving $m = 1, \frac{-1 \pm i\sqrt{3}}{2}$

Hence the general solution of the differential equation (1) is

$$y = Ae^x + e^{-\frac{1}{2}x}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right),$$ where A, B, C are three arbitrary constants.

4. Solve : $(D^3 - D)y = 0$

Solution Here the given differential equation is $(D^3 - D)y = 0$ ... (1)

Let $y = c_1e^{mx}$ $(c_1 \neq 0)$ be a trial solution of (1).

So, the auxiliary equation becomes as $m^3 - m = 0$ giving $m(m^2 - 1) = 0$

i.e., $m = 0, \pm 1$

Hence the general solution of the differential equation (1) is

$y = A + Be^x + Ce^{-x}$, where A, B, C are three arbitrary constants.

5. Solve : $(D^4 - D^2)y = 0$

Solution Here the given differential equation is $(D^4 - D^2)y = 0$ ... (1)

Let $y = c_1e^{mx}$ $[c_1 \neq 0]$ be a trial solution of (1).

Then the auxiliary equation becomes as $m^4 - m^2 = 0$

i.e., $m^2(m^2 - 1) = 0$ giving $m = 0, 0, 1, -1$

Hence the solution of the differential equation (1) is

$y = A + Bx + Ce^x + De^{-x}$, where A, B, C, D are four arbitrary constants.

6. Solve : $D(D - 1)^2y = e^{2x}$

Solution Here the given differential equation is $D(D - 1)^2y = e^{2x}$ ... (1)

The reduced equation of (1) is $D(D - 1)^2y = 0$ ... (2)

Let $y = c_1e^{mx}$ $[c_1 \neq 0]$ be a trial solution of the differential equation (2).

Then the auxiliary equation of (2) is $m(m - 1)^2 = 0$ giving $m = 0, 1, 1$

Since the roots of A.E. are real and multiple roots, so the complementary function of (1) is given by

$y_C = A + (B + Cx)e^x$; where A, B, C are three arbitrary constants.

Now particular integral $y_P = \frac{1}{D(D-1)^2}(e^{2x}) = \frac{e^{2x}}{2(2-1)^2} = \frac{e^{2x}}{2}$

Thus the required general solution of the given differential equation is $y = y_C + y_P$

$= A + (B + Cx)e^x + \frac{e^{2x}}{2}$, where A, B, C are three arbitrary constants.

7. Solve : $(D^4 + 4D^2 + 4)y = \sin x$

Solution Here the given differential equation is $(D^4 + 4D^2 + 4)y = \sin x$ ... (1)

The reduced equation of (1) is $(D^4 + 4D^2 + 4)y = 0$ (2)

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Let $y = c_1e^{mx}$ $(c_1 \neq 0)$ be a trial solution of (2).

Thus the auxiliary equation of (2) becomes $m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2 + 2)^2 = 0$ giving $m = \pm\sqrt{2}i, \pm\sqrt{2}i$

Thus the complementary function of (1) is

$y_C = (A + Bx)\cos\sqrt{2}x + (C + Dx)\sin\sqrt{2}x$, where A, B, C, D are four arbitrary constants

Now, P.I. $= \dfrac{1}{(D^2+2)^2}\sin x = \dfrac{\sin x}{(-1+2)^2} = \sin x$

Thus the required general solution of the given differential equation is $y = y_C + y_P$

i.e., $y = (A + Bx)\cos\sqrt{2}x + (C + Dx)\sin\sqrt{2}x$, where A, B, C, D are four arbitrary constants

8. Solve : $(D^3 - 3D^2 + 3D - 1)y = e^x$.

Solution The given differential equation of (1) is $(D^3 - 3D^2 + 3D - 1)y = e^x$ ... (1)

The reduced equation of (1) is $(D^3 - 3D^2 + 3D - 1)y = 0$

Let $y = c_1e^{mx}$ $(c_1 \neq 0)$ be a trial solution of (2). ... (2)

So, the auxiliary equation of (2) becomes as $(m^3 - 3m^2 + 3m - 1) = 0$

i.e., $(m-1)^3 = 0$, giving $m = 1, 1, 1$

Thus the C.F. of (1) is given by

$y_C = (A + Bx + Cx^2)e^x$, where A, B, C are three arbitrary constants.

Now particular integral is

$$y_P = \frac{1}{(D-1)^3}e^x = e^x\frac{1}{(D+1-1)^3} = e^x\frac{1}{D^3}1 = e^x.\frac{x^3}{6}$$

So, the general solution of (1) is $y = y_C + y_P$

i.e., $y = (A + Bx + Cx^2)e^x + \dfrac{x^3e^x}{6}$, where A, B, C are three orbitrary constants.

9. Show that if $l\dfrac{d^2\theta}{dt^2} + g\theta = 0$ and if $\theta = \alpha$ and $\dfrac{d\theta}{dt} = 0$ when $t = 0$, then $\theta = \alpha\cos\sqrt{\dfrac{g}{l}}\,t$ (g, l are positive).

[CP 1994, 1995]

Solution Here the given differential equation is $l\dfrac{d^2\theta}{dt^2} + g\theta = 0$

Let $\theta = ce^{mt}$ $(c \neq 0)$ be a trial solution of the differential equation (1). ... (1)

Then the A.E. of (1) becomes as $lm^2 + g = 0$, on solving $m = \pm i\sqrt{\dfrac{g}{l}}$, since g, l are both positive.

Since the roots of the A.E. are complex conjugate, so the general solution becomes $\theta = A\cos\sqrt{\dfrac{g}{l}}\,t + B\sin\sqrt{\dfrac{g}{l}}\,t$, where A and B are two arbitrary constants and can be determined from the given initial conditions.

Now, $\dfrac{d\theta}{dt} = -A\sqrt{\dfrac{g}{l}}\sin\sqrt{\dfrac{g}{l}}\,t + B\sqrt{\dfrac{g}{l}}\cos\sqrt{\dfrac{g}{l}}\,t$

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When $\theta = \alpha$, at $t = 0$, we have $A = \alpha$ and from $\frac{d\theta}{dt} = 0$ when $t = 0$,

we have $0 = B\sqrt{\frac{g}{l}}$ giving $B = 0$, since g, l are positive.

Therefore the general solution for particular case is $\theta = \alpha\cos\sqrt{\frac{g}{l}}\,t$. **[Proved]**

10. **If $\frac{d}{dx}(u(x)) = v(x)$, $\frac{d}{dx}(v(x)) = -u(x)$ and $u(0) = v(0) = 1$, find $u(x)$ and $v(x)$.** **[CP 1992]**

Solution Here it is given that $\frac{d}{dx}(u(x)) = v(x)$ and $\frac{d}{dx}(v(x)) = -u(x)$ ... (1)

Now $\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{dv}{dx} = -u(x)$ i.e., $\frac{d^2u}{dx^2} + u = 0$ ... (2)

and $\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d}{dx}(-u(x)) = -\frac{du}{dx} = -v$ i.e. $\frac{d^2v}{dx^2} + v = 0$ ... (3)

To solve (2), let $u = ce^{mx}$ ($c \neq 0$) be a trial solution.

Then A.E. becomes $m^2 + 1 = 0$ giving $m = \pm i$.

Therefore the general solution of (2) is $u(x) = A_1\cos x + B_1\sin x$

Similarly, the general solution of (3) is $v(x) = A_2\cos x + B_2\sin x$

where A_1, B_1, A_2, B_2 are arbitrary constants.

Again from $\frac{du}{dx} = v(x)$ and $\frac{dv}{dx} = -u(x)$, we have $\frac{du}{dx} = v(x) = B_1\cos x - A_1\sin x$ and

$\frac{dv}{dx} = -u(x) = -A_2\sin x + B_2\cos x$. From $v(0) = 1$, we have $v(0) = A_1\cos(0) + B_1\sin(0)$

i.e., $A_1 = 1$ and $-u(0) = -A_2\sin(0) + B_2\cos(0)$ *i.e.*, $B_2 = -1$.

Similarly, from $v(0) = 1$, we have, $1 = A_2$ and $B_1 = 1$.

Therefore, the required solution is $u(x) = \cos x + \sin x$ and $v(x) = \cos x - \sin x$.

11. **Find a particular integral of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$.** **[CP 2010]**

Solution Here the given differential equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ ... (1)

Introducing the operator $D \equiv \frac{d}{dx}$, we have $(D^2 - 3D + 2)y = e^{3x}$ ... (2)

We have from (2), P.I. is $y_P = \frac{1}{D^2 - 3D + 2}e^{3x} = \frac{1}{9 - 9 + 2}e^{3x} = \frac{1}{2}e^{3x}$

12. **Find the particular integral of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$.** **[CP 2001, 2009]**

Solution Here the given differential equation is $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$ ... (1)

Introducing the operator $D \equiv \frac{d}{dx}$, we have $(D^2 + 4D + 3)y = e^{-3x}$ ... (2)

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Thus y_P, the P.I. of (1) is given by

$$y_P = \frac{1}{D^2+4D+3}(e^{-3x}) = \frac{1}{D+3}\left(\frac{1}{D+1}e^{-3x}\right)$$

$$= \frac{1}{D+3}\left(\frac{e^{-3x}}{-2}\right) \quad \left[\text{since } \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}\right]$$

$$= \frac{e^{-3x}}{-2}\frac{1}{(D-3)+3}(1) \quad \left[\text{since } \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$$

$$= -\frac{e^{-3x}}{2}\frac{1}{D}(1) = -\frac{xe^{-3x}}{2}$$

Hence the required particular integral is $-\frac{xe^{-3x}}{2}$.

13. **Find the particular integral of the following differential equation**

(a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$

(b) $(D^2 + 9)y = \cos\left(x + \frac{\pi}{6}\right)$ **(c)** $(D-1)^2 = xe^x\sin x$ [CP 2

Solution (a) Introducing the operator $D \equiv \frac{d}{dx}$,

the given equation can be written as $(D^2 - 2D + 5)y = 10\sin x$ ...(1)

If y_P be a P.I., then we have from (1),

$$y_P = \frac{1}{D^2-2D+5}(10\sin x) = 10\frac{D^2+2D+5}{(D^2+5-2D)(D^2+5+2D)}(\sin x)$$

$$= 10\frac{D^2+2D+5}{(D^2+5)^2-4D^2}(\sin x) = 10\frac{D^2+2D+5}{(-1+5)^2-4(-1)}(\sin x)$$

$$= \frac{10}{20}(D^2+2D+5)(\sin x) = \frac{1}{2}(-\sin x + 2\cos x + 5\sin x)$$

$= 2\sin x + \cos x$, which is the required particular integral.

■ **Alternative Method**

$$y_P = \frac{1}{D^2-2D+5}10\sin x = 10\frac{1}{-1^2-2D+5}\sin x$$

$$= 10\frac{1}{4-2D}\sin x = -5\frac{1}{D-2}\sin x = -5\frac{D+2}{D^2-4}\sin x = -5\frac{\cos x + 2\sin x}{-1^2-4}$$

$= 2\sin x + \cos x$, which is the required particular integral.

(b) Let y_P be the P.I., we have $y_P = \frac{1}{D^2+9}\cos\left(x+\frac{\pi}{6}\right) = \frac{\cos\left(x+\frac{\pi}{6}\right)}{-1^2+9} = \frac{\cos\left(x+\frac{\pi}{6}\right)}{8}$

(c) Let y_P be the P.I., we have $y_P = \frac{1}{(D-1)^2}xe^x\sin x = e^x\frac{1}{(D+1-1)^2}x\sin x = e^x\frac{1}{D^2}x\sin x$ ...(1)

Now, $\frac{1}{D}x\sin x = \int x\sin x\,dx = -x\cos x + \sin x$

Again, $\frac{1}{D^2}x\sin x = \int(-x\cos x + \sin x)dx = -(x\sin x + 2\cos x)$

Therefore, $y_P = -e^x(x\sin x + 2\cos x)$.

14. Solve : $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = xe^x$ [CP 2015]

Solution Introducing the operator $D \equiv \frac{d}{dx}$, the given differential equation can be written as

$$(D^2 + D - 6)y = xe^x \quad \text{... (1)}$$

The reduced form of the equation is $(D^2 + D - 6)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. becomes $m^2 + m - 6 = 0$

i.e., $(m + 3)(m - 2) = 0$ and on solving $m = 2, -3$

Since the roots of the auxiliary equation are real and unequal, therefore the C.F. of the given differential equation is $y_C = Ae^{2x} + Be^{-3x}$, where A, B are two arbitrary constants.

To find P.I., we have from (1),

$$y_P = \frac{1}{D^2+D-6}xe^x = e^x\frac{1}{(D+1)^2+(D+1)-6}(x) \left[\text{since } \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$$

$$= e^x\frac{1}{D^2+3D-4}(x) = e^x\frac{1}{-4}\left(1-\frac{D^2+3D}{4}\right)^{-1}(x) = -\frac{e^x}{4}\left(1+\frac{D^2+3D}{4}+\cdots\right)(x)$$

$$= -\frac{e^x}{4}\left(x+\frac{3}{4}\right) = -\frac{e^x}{16}(4x+3)$$

Therefore, the general solution is $y = y_C + y_P$

i.e., $y = Ae^{2x} + Be^{-3x} - \frac{e^x}{16}(4x + 3)$, where A and B are two arbitrary constants.

15. Solve : $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^3e^{2x}$. [CP 2015]

Solution Introducing the operator $D \equiv \frac{d}{dx}$, the given differential equation can be written as

$$(D^2 - 4D + 4)y = x^3e^{2x} \quad \text{... (1)}$$

Now the reduced form of the differential equation (1) is $(D^2 - 4D + 4)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. is $m^2 - 4m + 4 = 0$

i.e., $(m - 2)^2 = 0$, solving $m = 2, 2$

Therefore C.F. is $y_C = (A + Bx)e^{2x}$, where A, B are two arbitrary constants.

To find P.I., we have from (1),

$$y_P = \frac{1}{D^2-4D+4}(x^3e^{2x}) = \frac{1}{(D-2)^2}(e^{2x}x^3)$$

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$$= e^{2x}\frac{1}{(D+2-2)^2}(x^3) \quad \left[\text{since } \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$$

$$= e^{2x}\frac{1}{D^2}(x^3) = e^{2x}\frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x}\left(\frac{x^5}{20}\right) = \frac{x^5e^{2x}}{20}$$

Hence the general solution of the given differential equation is

$y = y_C + y_P = \left(A + Bx + \frac{x^5}{20}\right)e^{2x}$, where A and B are two arbitrary constants.

16. Solve : $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x + x^2$ [CP 2008]

Solution Introducing the operator $D \equiv \frac{d}{dx}$, the given differential equation can be written as

$(D^2 + 2D + 1)y = 2x + x^2$ ... (1)

The reduced equation of (1) is $(D^2 + 2D + 1)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. is $m^2 + 2m + 1 = 0$ giving $(m+1)^2 = 0$ and on solving we have, $m = -1, -1$.

Thus the C.F. is $y_C = (A + Bx)e^{-x}$, where A, B are two arbitrary constants.

To find P.I., we have from (1)

$$y_P = \frac{1}{D^2+2D+1}(2x + x^2) = (1 + 2D + D^2)^{-1}(2x + x^2)$$

$$= \{1 - (2D + D^2) + (2D + D^2)^2 + \ldots\}(2x + x^2)$$

$$= \{1 - (2D + D^2) + 4D^2 + \ldots\}(2x + x^2)$$

$$= (2x + x^2) - 2(2 + 2x) + 3 \times 2 = x^2 - 2x + 2$$

Hence the required general solution is

$y = y_C + y_P = (A + Bx)e^{-x} + x^2 - 2x + 2$, where A and B are two arbitrary constants.

■ ***Alternative Method to find P.I.***

$$y_P = \frac{1}{(1+D)^2}(2x + x^2) = (1 + D)^{-2}(2x + x^2) = (1 - 2D + 3D^2)(2x + x^2)$$

$$= 2x + x^2 - 2(2 + 2x) + 3\cdot2 = x^2 - 2x + 2$$

17. Solve : $(D^2 + a^2)y = \cos ax + \cos bx$, $D \equiv \frac{d}{dx}$ [CP 1999]

Solution Here the given differential equation is

$(D^2 + a^2)y = \cos ax + \cos bx$, $D \equiv \frac{d}{dx}$ ... (1)

The reduced equation of (1) is $(D^2 + a^2)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. is $m^2 + a^2 = 0$ giving $m = \pm ia$.

Since the roots of A.E. are complex conjugate, so the C.F. is

$y_C = A\cos ax + B\sin ax$, where A, B are two arbitrary constants.

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To find P.I., we have from (1), $y_P = \frac{1}{D^2+a^2}(\cos ax + \cos bx)$ ... (3)

Now, $\frac{1}{D^2+a^2}(\cos bx) = \frac{1}{-b^2+a^2}\cos bx$ $\left[\text{since } \frac{1}{f(D^2)}\cos px = \frac{1}{f(-p^2)}\cos px \text{ for } f(-p^2) \neq 0\right]$

$= \frac{1}{a^2-b^2}\cos bx$

Again, $\frac{1}{D^2+a^2}(\cos ax)$ = Real part of $\frac{1}{D^2+a^2}(e^{iax})$ [since $e^{iax} = \cos ax + i\sin ax$]

= Real part of $e^{iax}\frac{1}{(D+ia)^2+a^2}(1)$ $\left[\text{since } \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$

= Real part of $e^{iax}\frac{1}{D^2+2iaD}(1)$

= Real part of $e^{iax}\frac{1}{D(2ia+D)}(1)$

= Real part of $e^{iax}\frac{1}{D}\frac{1}{2ia}\left(1+\frac{D}{2ia}\right)^{-1}(1)$

= Real part of $e^{iax}\left(-\frac{i}{2a}\right)\frac{1}{D}(1)$

= Real part of $(\cos ax + i\sin ax)\left(-\frac{i}{2a}\right)(x)$

$= \frac{x\sin ax}{2a}$

■ **Alternative Method for P.I.**

$\frac{1}{D^2+a^2}(\cos ax) = x\frac{1}{2D}\cos ax$

$= \frac{x}{2}\left(\frac{\sin ax}{a}\right)$

$= \frac{x\sin ax}{2a}$

Thus we have from (3), $y_P = \frac{x}{2a}\sin ax + \frac{\cos bx}{a^2-b^2}$

Hence the general solution is $y = y_C + y_P = A\cos ax + B\sin ax + \frac{x}{2a}\sin ax + \frac{\cos bx}{a^2-b^2}$, where A, B are two arbitrary constants.

[CP 1994]

18. **Solve : $\frac{d^2y}{dx^2} + a^2y = \sin ax$.**

Solution Introducing the operator $D \equiv \frac{d}{dx}$, the given differential equation can be written as

$(D^2+a^2)y = \sin ax$ ... (1)

The reduced form of (1) is $(D^2+a^2)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. is $m^2 + a^2 = 0$, solving $m = \pm ia$

Since the roots of A.E. are complex conjugate, therefore the C.F. is $y_C = A\cos ax + B\sin ax$,

where A, B are arbitrary constants.

To find P.I., we have from (1),

$y_P = \frac{1}{D^2+a^2}(\sin ax)$ = Imaginary part of $\frac{1}{D^2+a^2}(e^{iax})$ [since $e^{iax} = \cos ax + i\sin ax$]

= Imaginary part of $e^{iax}\frac{1}{(D+ia)^2+a^2}(1)$

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$= \text{Imaginary part of } e^{iax}\dfrac{1}{D^2+2iaD}(1)$

$= \text{Imaginary part of } e^{iax}\dfrac{1}{D(2ia+D)}(1)$

$= \text{Imagianry part of } e^{iax}\dfrac{1}{D}\dfrac{1}{2ia}\left(1+\dfrac{D}{2ia}\right)^{-1}(1)$

$= \text{Imaginary part of } e^{iax}\left(\dfrac{-i}{2a}\right)\dfrac{1}{D}\left(1-\dfrac{D}{2ia}+\cdots\right)(1)$

$= \text{Imaginary part of } e^{iax}\left(-\dfrac{i}{2a}\right)\dfrac{1}{D}(1)$

$= \text{Imaginary part of } (\cos ax + i\sin ax)\left(-\dfrac{i}{2a}\right)(x)$

$= -\dfrac{x\cos ax}{2a}$

■ **Alternative Method for P.I.**

$y_P = \dfrac{1}{D^2+a^2}\sin ax = x\dfrac{1}{2D}\sin ax$

$= \dfrac{x}{2}\dfrac{-\cos ax}{a} = -\dfrac{x\cos ax}{2a}$

Hence the required general solution is $y = y_C + y_P$

i.e., $y = A\cos ax + B\sin ax - \dfrac{x}{2a}\cos ax = \left(A - \dfrac{x}{2a}\right)\cos ax + B\sin ax$

where A, B are two arbitrary constants.

19. **Solve : $(D^2 - 7D + 12)y = \log x$, $D \equiv \dfrac{d}{dx}$.**

Solution Here the given differential equation is

$$(D^2 - 7D + 12)y = \log x,\ D \equiv \frac{d}{dx} \quad \text{... (1)}$$

The corresponding homogeneous equation of (1) is $(D^2 - 7D + 12)y = 0$ ...(2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. becomes $m^2 - 7m + 12 = 0$, solving $m = 3, 4$

Since the two roots of A.E. are real and unequal, therefore the C.F. of (1) is

$y_C = Ae^{3x} + Be^{4x}$, where A, B are two arbitrary constants.

To find P.I., we have from (1)

$$y_P = \frac{1}{D^2-7D+12}(\log x) = \frac{1}{(D-3)(D-4)}(\log x) = \frac{(D-3)-(D-4)}{(D-3)(D-4)}\log x$$

$$= \left(\frac{1}{D-4} - \frac{1}{D-3}\right)\log x = \frac{1}{D-4}(\log x) - \frac{1}{D-3}(\log x)$$

$$= e^{4x}\int e^{-4x}(\log x)dx - e^{3x}\int e^{-3x}(\log x)\,dx \quad \left[\text{Since } \frac{1}{D-m}V = e^{mx}\int e^{-mx}V\,dx\right]$$

Hence the general solution is $y = y_C + y_P$

i.e., $y = Ae^{3x} + Be^{4x} + e^{4x}\int e^{-4x}\log x\,dx - e^{3x}\int e^{-3x}\log x\,dx$

where A, B are two arbitrary constants.

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20. Solve : $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$.

Solution Here the given equation is $(D^2 + 3D + 2)y = e^{e^x}$, $D \equiv \dfrac{d}{dx}$ ... (1)

The reduced homogeneous equation of (1) is $(D^2 + 3D + 2)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. is $m^2 + 3m + 2 = 0$, on solving $m = -1, -2$.

Since the roots of the A.E. are real and unequal,

therefore the C.F. of (1) is $y_C = Ae^{-x} + Be^{-2x}$, where A and B are two arbitrary constants.

To find P.I., we have from (1)

$$y_P = \frac{1}{(D^2+3D+2)}e^{e^x} = \frac{(D+2)-(D+1)}{(D+2)(D+1)}e^{e^x} = \left[\frac{1}{D+1} - \frac{1}{D+2}\right]e^{e^x} = \frac{1}{D+1}e^{e^x} - \frac{1}{D+2}e^{e^x}$$

$$= e^{-x}\int e^x e^{e^x}dx - e^{-2x}\int e^{2x}e^{e^x}dx \text{ [since } \frac{1}{D-m}f(x) = e^{mx}\int e^{-mx}f(x)\,dx]$$

Now, $\int e^x e^{e^x}dx = \int e^u du$, if $u = e^x$

$= e^u = e^{e^x}$

and $\int e^{2x}e^{e^x}dx = \int ue^u\,du$, if $u = e^x$

$= ue^u - e^u = e^x e^{e^x} - e^{e^x}$

Thus $y_p = e^{-x}e^{e^x} - e^{-2x}(e^x e^{e^x} - e^{e^x}) = e^{-2x}e^{e^x}$

Hence the general solution is $y = y_C + y_P$

i.e., $y = Ae^{-x} + Be^{-2x} + e^{-2x}e^{e^x}$, where A and B are two arbitrary constants.

21. Solve : $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{-x} + \cos x$ [CH 2008]

Solution Here the given differential equation is

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \cos x \quad \text{... (1)}$$

The reduced equation of (1) is $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$ ... (2)

Let $y = c_1e^{mx}$ $(c_1 \neq 0)$ be a trial solution of (2). Then the auxiliary equation of (2) becomes

$m^3 + 2m^2 + m = 0$ *i.e.*, $m(m+1)^2 = 0$ giving $m = 0, -1, -1$

Thus the complementary function of (1) is

$y_C = A + (B + Cx)e^{-x}$, where A, B and C are three arbitrary constants.

To find P.I., let $y = y_P$ be a particular integral of (1).

Then introducing the operator $D \equiv \dfrac{d}{dx}$, we have from (1),

$$y_P = \frac{1}{D^3+2D^2+D}(e^{-x} + \cos x) = \frac{1}{D(D+1)^2}(e^{-x}) + \frac{1}{D(D+1)^2}(\cos x)$$

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$= \frac{1}{(D+1)^2}(-e^{-x}) + \frac{1}{D^2+2D+1}(\sin x)$ [since $\frac{1}{D}(e^{-x}) = -e^{-x}$ and $\frac{1}{D}(\cos x) = \sin x$]

$= -e^{-x}\frac{1}{(D-1+1)^2}(1) + \frac{1}{-1+2D+1}(\sin x)$

$= -e^{-x}\frac{1}{D^2}(1) + \frac{1}{2}\cdot\frac{1}{D}(\sin x) = -\frac{x^2}{2}e^{-x} - \frac{1}{2}\cos x$

Hence the required general solution of the given equation is

$$y = y_C + y_P = A + \left(B + Cx - \frac{x^2}{2}\right)e^{-x} - \frac{1}{2}\cos x,$$

where A, B, C are three arbitrary constants.

22. **Solve : $D^2(D^2 + D + 1)y = x^2$** [CH 19..]

Solution Here the given differential equation is $D^2(D^2 + D + 1)y = x^2$ ...(1)

The reduced equation of (1) is thus $D^2(D^2 + D + 1)y = 0$ ...(2)

Let $y = c_1e^{mx}$ $(c_1 \neq 0)$ be a trial solution of (2). Then the auxiliary equation of (2) becomes

$$m^2(m^2 + m + 1) = 0 \text{ giving } m = 0, 0, -\frac{1}{2}, \pm\frac{\sqrt{3}}{2}i$$

Thus the complementary function of (1) is given by :

$y_C = A + Bx + e^{-\frac{x}{2}}\left(C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right)$, where A, B, C, D are four arbitary constants

To find particular intergral, let $y = y_P$ be a particular integral of (1).

Then $y_P = \frac{1}{D^2(D^2+D+1)}(x^2) = \frac{1}{D^2+D+1}\left(\frac{x^4}{12}\right)$ [since $\frac{1}{D}(x^2) = \frac{x^3}{3}$ and $\frac{1}{D}\left(\frac{x^3}{3}\right) = \frac{x^4}{12}$]

$= \frac{1}{12}[1 + (D + D^2)]^{-1}(x^4)$

$= \frac{1}{12}[1 - (D + D^2) + (D + D^2)^2 - (D + D^2)^3 + (D + D^2)^4 \ldots](x^4)$

$= \frac{1}{D^2}[1 + (D + D^2)]^{-1}(x^2) = \frac{1}{D^2}[1 - (D + D^2) + (D + D^2)^2 - \ldots](x^2)$

$= \frac{1}{D^2}[1 - (D + D^2) + D^2 + 2D^3 + \ldots](x^2)$

$= \frac{1}{D^2}(1 - D + 2D^3 + \ldots)(x^2)$

$= \frac{1}{D^2}(x^2 - 2x) = \frac{x^4}{12} - \frac{x^3}{3}$

Thus the required general solution of the given differential equation is

$$y = y_C + y_P = A + Bx + e^{-\frac{x}{2}}\left(C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right) + \frac{x^4}{12} - \frac{x^3}{3},$$

where A, B, C and D are four arbitrary constants.

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23. Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$ given that when $y = 0$, $\frac{dy}{dx} = 2$ at $x = 0$. [CP 1994]

Solution Introducing the operator $D \equiv \frac{d}{dx}$, we can write the given differential equation as

$(D^2 - 2D + 5) = 10\sin x$ ... (1)

The reduced form of (1) is $(D^2 - 2D + 5)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then the A.E. will be $(m^2 - 2m + 5) = 0$, solving we have $m = \frac{1}{2}(2 \pm \sqrt{4-20}) = 1 \pm 2i$.

Therefore the C.F. is $y_C = e^x(A\cos 2x + B\sin 2x)$, where A, B are arbitrary constants.

To find P.I., we have from (1),

$$y_P = \frac{1}{D^2 - 2D + 5}(10\sin x) = 10\frac{D^2 + 2D + 5}{(D^2 + 5 - 2D)(D^2 + 5 + 2D)}(\sin x)$$

$$= 10\frac{D^2 + 2D + 5}{(D^2 + 5)^2 - 4D^2}(\sin x) = 10(D^2 + 2D + 5)\left(\frac{1}{(D^2 + 5)^2 - 4D^2}(\sin x)\right)$$

$$= 10(D^2 + 2D + 5)\frac{1}{(-1 + 5)^2 - 4(-1)}(\sin x) = \frac{1}{2}(D^2 + 2D + 5)(\sin x)$$

$$= \frac{1}{2}(-\sin x + 2\cos x + 5\sin x) = \cos x + 2\sin x$$

Thus the general solution is $y = y_C + y_P$

i.e., $y = e^x(A\cos 2x + B\sin 2x) + \cos x + 2\sin x$ ... (3)

Now $\frac{dy}{dx} = e^x(-2A\sin 2x + 2B\cos 2x + A\cos 2x + B\sin 2x) - \sin x + 2\cos x$

$= e^x\{(B - 2A)\sin 2x + (A + 2B)\cos 2x\} - \sin x + 2\cos x$ ... (4)

Using the initial condition $y = 0$ at $x = 0$, from (3),

$A + 1 = 0$ *i.e.*, $A = -1$

Using the condition $-\frac{dy}{dx} = 2$ at $x = 0$, from (4),

$(A + 2B) + 2 = 2$ *i.e.*, $2B = 1$ [since $A = -1$], giving $B = \frac{1}{2}$

Hence the required solution is $y = e^x(-\cos 2x + \frac{1}{2}\sin 2x) + \cos x + 2\sin x$

■ Alternative Method to find P.I.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 5}(10\sin x)$$

$$= 10\frac{1}{-1^2 - 2D + 5}\sin x = 5\frac{1}{2 - D}\sin x$$

$$= 5\frac{2 + D}{4 - D^2}\sin x = 5\frac{(2\sin x + \cos x)}{4 - (-1^2)} = 2\sin x + \cos x$$

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24. Solve : $\frac{d^2y}{dx^2} + y = \sin 2x$, given that when $x = 0$ both $y = 0$ and $\frac{dy}{dx} = 0$

[CP 2012]

Solution Introducing the differential operator $D \equiv \frac{d}{dx}$, the given equation can be written as

$(D^2 + 1)y = \sin 2x$ ... (1)

The reduced form of (1) is $(D^2 + 1)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2).

Then the A.E. is $m^2 + 1 = 0$ giving $m = \pm i$.

Since the roots of the A.E. are complex conjugate,

then the C.F. is $y_C = A\cos x + B\sin x$; A, B are two arbitrary constants.

To find P.I., we have from (1)

$$y_P = \frac{1}{D^2+1}(\sin 2x) = \frac{1}{-4+1}(\sin 2x) \text{ [since } \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax]$$

$$= -\frac{\sin 2x}{3}$$

Thus the general solution is $y = y_C + y_P$

i.e., $y = A\cos x + B\sin x - \frac{\sin 2x}{3}$

where A and B are two arbitrary constants. ... (3)

To determine A and B, let us use the initial conditions.

From (3) we have, $\frac{dy}{dx} = -A\sin x + B\cos x - \frac{2\cos 2x}{3}$ ... (4)

Using the initial condition at $x = 0$, $y = 0$, from (3), we have $A = 0$

Again using the initial condition at $x = 0$, $\frac{dy}{dx} = 0$, from (4) we have, $B - \frac{2}{3} = 0$ *i.e.*, $B = \frac{2}{3}$

Thus from (3), the required solution is

$$y = \frac{2}{3}\sin x - \frac{\sin 2x}{3} = \frac{1}{3}(2\sin x - \sin 2x)$$

25. **Find P.I. of $(D^2 + 4D + 5)y = 2\cosh x$.**

Solution Let y_P be a P.I. of the above differential equation

Therefore, $y_P = \frac{1}{D^2+4D+5}2\cosh x = \frac{1}{D^2+4D+5}2\left(\frac{e^x+e^{-x}}{2}\right)\left[\because \cosh x = \frac{e^x+e^{-x}}{2}\right]$

$$= \frac{1}{D^2+4D+5}e^x + \frac{1}{D^2+4D+5}e^{-x}$$

$$= \frac{e^x}{1+4\cdot 1+5} + \frac{e^{-x}}{(-1)^2+4(-1)+5} = \frac{e^x}{10} + \frac{e^{-x}}{2}$$

Hence the required P.I. is $\frac{e^x}{10} + \frac{e^{-x}}{2}$

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Exercise

1. Solve : $2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 12x = 0$
2. Find the C.F. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = xe^x$

- **Solve the following (3-27) :**

3. $(D^3 + 1)y = 0$
4. $(D^4 - D)y = 0$
5. $(D^3 + D^2 - 2D)y = 0$
6. $(D - 2)^3(D^2 + 4)y = 0$
7. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = x^2$ [CP 1990]
8. $\frac{d^2y}{dx^2} - y = e^x + e^{-x}$ [CP 1992]
9. $\frac{d^2y}{dx^2} + 4y = \sin x$ [CP 1995, 2000]
10. $(D - 1)^2y = xe^x\sin x$ [NBU 1986]
11. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x\sin x$ [BH 2006]
12. $(D^2 + 1)y = 3\cos^2 x + 2\sin^3 x$ [CP 2014]
13. $\frac{d^2y}{dx^2} + a^2y = \sec ax$ [CH 1995, 1999]
14. $(D - 2)^2y = \cosh x$
15. $(D^3 - 4D)y = e^x + e^{-x}$
16. $(D^3 - 2D + 4)y = e^x\sin\frac{x}{2}$
17. $(D^3 + 6D^2 + 12D + 8)y = e^{-2x}$
18. $(D^3 - D^2)y = x$
19. $(D^4 + 1)y = \cos x$
20. $(D^3 - 3D^2 + 3D - 1)y = xe^x + e^x$ [BH 1981]
21. $(D^3 - 7D - 6)y = (1 + x)e^{2x}$ [VH 1987]
22. $\frac{d^3y}{dx^3} + y = e^{2x}\sin x + e^{\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$ [CH 1992]
23. $\frac{d^4y}{dx^4} - y = e^x\cos x$ [BH 1991]
24. $(D - 1)^2(D^2 + 1)^2y = e^x + x$ [CH 1983]
25. $(D^5 - D)y = e^x + \sin x - x$ [CH 1981]
26. $(D^2 - 2D + 2)y = 5\cos x$ given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.
27. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^x$ given that $y = 1$, $\frac{dy}{dx} = 1$, when $x = 0$ [CP 2001]

Answers

1. $x = Ae^{\frac{3t}{2}} + Be^{-4t}$
2. $y = C_1 + C_2e^{-\frac{x}{2}}$
3. $y = Ae^{-x} + e^{\frac{x}{2}}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right)$
4. $y = A + Be^x + e^{-\frac{x}{2}}\left(C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right)$
5. $y = A + Be^{-2x} + Ce^x$
6. $y = (A + Bx + Cx^2)e^{2x} + D\cos 2x + E\sin 2x$
7. $y = e^x(A\cos\sqrt{3}\,x + B\sin\sqrt{3}\,x) + \frac{1}{4}(x^2 + x)$
8. $y = Ae^x + Be^{-x} + \frac{x}{2}(e^x - e^{-x})$
9. $y = A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$
10. $y = (A + Bx)e^x - (2\cos x + x\sin x)e^x$
11. $y = A + Be^{2x} - \frac{1}{2}e^x\sin x$
12. $y = A\cos x + B\sin x + \frac{3}{2} - \frac{1}{2}\cos 2x - \frac{3}{4}x\cos x + \frac{1}{16}\sin 3x$
13. $y = A\cos ax + B\sin ax + \frac{1}{a}\left(x\sin ax + \frac{1}{a}\cos ax\log\cos ax\right)$

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14. $y = (A + Bx)e^{2x} + \frac{x^2}{4}e^{2x} + \frac{e^{-2x}}{32}$

15. $y = A + Be^{2x} + Ce^{-2x} - \frac{1}{3}x(e^x + e^{-x})$

16. $y = Ae^{-2x} + e^x(B\cos x + B\sin x) - \frac{16}{111}e^x\left(\frac{1}{2}\cos\frac{1}{2}x - 3\sin\frac{x}{2}\right)$

17. $y = e^{-2x}(A + Bx + Cx^2) + \frac{x^3}{6}e^{-2x}$

18. $y = A + Bx + Ce^x - \left(\frac{x^3}{6} + \frac{x^2}{2}\right)$

19. $y = e^{\frac{1}{\sqrt{2}}x}(A\cos x + B\sin x) + e^{-\frac{1}{\sqrt{2}}x}(C\cos x + D\sin x) + \frac{1}{2}\cos x$

20. $y = (C_1 + C_2x + C_3x^2)e^x + \frac{1}{24}e^x(x^4 + 4x^3)$

21. $y = Ae^{-x} + Be^{-2x} + Ce^{3x} - \frac{1}{12}e^{2x}\left(x + \frac{17}{12}\right)$

22. $y = Ae^{-x} + \left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right)e^{\frac{x}{2}} + \frac{e^{2x}}{130}(3\sin x - 11\cos x) - \frac{1}{6}xe^{\frac{x}{2}}\left(\sin\frac{\sqrt{3}}{2}x + \sqrt{3}\cos\frac{\sqrt{3}}{2}x\right)$

23. $y = Ae^x + Be^{-x} + C\sin x + D\cos x - \frac{1}{5}e^x\cos x$

24. $y = (A + Bx)e^x + (C + Dx)\cos x + (E + Fx)\sin x + \frac{1}{8}x^2e^x + x + 2$

25. $y = A + Be^x + Ce^{-x} + D\cos x + E\sin x + \frac{1}{4}xe^x + \frac{1}{4}x\sin x + \frac{1}{2}x^2$

26. $y = (3\sin x - \cos x)e^x + \cos x - 2\sin x$

27. $y = e^x$

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8 CHAPTER

Homogeneous Linear Differential Equation

Introduction

A differential equation of order n of the following form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = \phi(x) \qquad \ldots (1)$$

where a_1, a_2, a_{n-1}, a_n all are constants and $\phi(x)$ is either a function of independent variable x or a constant is known as **Cauchy-Euler differential equaion**. Further the equation (1) is also known as homogeneous linear differential equation of order n. First we describe the solution procedure of second order Cauchy-Euler differential equation and then discuss the technique to solve higher order Cauchy-Euler equation.

8.1 Solution Procedure of Second Order Cauchy-Euler Equation

The general form of the second order homogeneous differential equation is;

$$x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = \phi(x), \qquad \ldots (2)$$

where a_1, a_2 are two constants and $\phi(x)$ is a function of x.

To solve (2), let us substitute $x = e^z$ (*i.e.*, $z = \log x$)

Then $\frac{dx}{dz} = e^z = x$

$$\text{Now } \frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = x\frac{dy}{dx} \quad i.e., \; x\frac{dy}{dx} = \frac{dy}{dz} \qquad \ldots (3)$$

$$\text{Also, } \frac{d^2 y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dx}{dz} = \frac{d}{dx}\left(x\frac{dy}{dx}\right)x = x\left\{x\frac{d^2 y}{dx^2} + \frac{dy}{dx}\right\} = x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx}$$

$$i.e., \; x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - x\frac{dy}{dx} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} \qquad \ldots (4)$$

With the help of (3) and (4), equation (2) can be written as :

$$\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) + a_1 \frac{dy}{dz} + a_2 y = \phi(e^z) \quad i.e., \; \frac{d^2 y}{dz^2} + (a_1 - 1)\frac{dy}{dz} + a_2 y = \phi(e^z) \qquad \ldots (5)$$

Equation (5) in a second order linear differential equation with constant co-efficient and this can be easily solved by finding C.F. and P.I.

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Notes

1. Introducing the operator $D \equiv \frac{d}{dx}$ and $D' \equiv \frac{d}{dz}$ the relation between them can be established as $xD \equiv D'$ and $x^2D^2 \equiv D'^2 - D' \equiv D'(D'-1)$.
   Similarly, we may write,
   $x^3D^3 \equiv D'(D'-1)(D'-2)$
   $x^4D^4 \equiv D'(D'-1)(D'-2)(D'-3)$
   ....
   ....
   $x^nD^n \equiv D'(D'-1)(D'-2)(D'-3) \ldots (D'-n+1)$
2. If a differential equation is of third and higher order with Cauchy-Euler form, then to solve it we need to first reduce it as a differential equation with constant coefficient with same order. Then it can be solved by finding complementary function and particular integral.

Illustration-1 : Solve : $(x^3D^3 + 3x^2D^2 + xD + 1)y = x;\ D \equiv \frac{d}{dx}$

Solution : Here the given differential equation is

$$(x^3D^3 + 3x^2D^2 + xD + 1)y = x \quad \ldots (6)$$

To solve (6), we substitute $z = \log x$, with this substitution, we have

$$xD \equiv D',\ x^2D^2 \equiv D'(D'-1) \text{ and } x^3D^3 \equiv D'(D'-1)(D'-2), \text{ where } D' \equiv \frac{d}{dz}$$

Then the equation (6) reduces to :

$$(D'(D'-1)(D'-2) + 3D'(D'-1) + D' + 1)y = e^z \quad i.e.,\ (D'^3 + 1)y = e^z$$

Clearly, the C.F. of above equation is

$y_C = Ae^{-z} + e^{z/2}\left(B\cos\frac{\sqrt{3}}{2}z + C\sin\frac{\sqrt{3}}{2}z\right)$, where A, B, C are three arbitrary constants and

P.I. is $y_P = \frac{1}{(D'^3+1)}(e^z) = \frac{1}{2}e^z$

Thus the general solution becomes : $y = \frac{A}{x} + \sqrt{x}\left(B\cos\left(\frac{\sqrt{3}}{2}\log x\right) + C\sin\left(\frac{\sqrt{3}}{2}\sin x\right)\right) + \frac{x}{2}$

8.2 Equation Reducible to Homogeneous Linear From

First we consider a second order linear differential equation of the following from :

$$(a+bx)^2\frac{d^2y}{dx^2} + a_1(a+bx)\frac{dy}{dx} + a_2 = \phi(x) \quad \ldots(7)$$

The from (7) in known as the **Legendre's equation** and it can be reduced to the homogeneous linear form by substituting $u = a + bx$ and then substituting $u = e^z$ that equation can be reduced to second order linear equation with constant co-efficient which can be solved easily. Further to evaluate these types of equations sometimes we directly substitute $e^z = a + bx$ and then solve. Similarly we can solve a differential equation of third and higher order which can be reduced to homogeneous form or Cauchy-Euler form. In a similar manner we can find general solution of third and higher order differential equation which can be reduced to Cauchy-Euler form.

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Illustrative Examples

1. **Find the particular integral of the differential equation $(x^2D^2 - xD + 1)y = 2\log x$** [CP 1988]

Solution Here the given differential equation is $(x^2D^2 - xD + 1)y = 2\log x$ ... (1)

Putting $x = e^z$ i.e., $z = \log x$

We have $x\dfrac{dy}{dx} = \dfrac{dy}{dz}$ and $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$

$\therefore$ The given differential equation (1) reduces to $\dfrac{d^2y}{dz^2} - 2\dfrac{dy}{dz} + y = 2z$

i.e., $(D'^2 - 2D' + 1)y = 2z$, where $D' = \dfrac{d}{dz}$

$$\text{P.I.} = \frac{1}{D'^2 - 2D' + 1}(2z) = 2[1 + (D'^2 - 2D')]^{-1}(z) = 2[1 - (D'^2 - 2D')](z)$$

$$= 2[z - 0 + 2] = 2[\log x + 2] = 2\log x + 4$$

2. **Solve : $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x.$** [CP 1985, 1991, 1994, 2000, 2012]

Solution Introducing $D \equiv \dfrac{d}{dx}$, the given differential equation can be written as

$$(x^2D^2 - xD + 1)y = \log x \quad \text{... (1)}$$

To solve (1), let us substitute $z = \log x$

Then $\dfrac{dz}{dx} = \dfrac{1}{x}$ giving $\dfrac{dx}{dz} = x.$

Now, $\dfrac{dy}{dz} = \dfrac{dy}{dx}\cdot\dfrac{dx}{dz} = x\dfrac{dy}{dx}$ i.e., $xDy = D'y$, where $D' = \dfrac{d}{dz}$ ... (2)

Also, $\dfrac{d^2y}{dz^2} = \dfrac{d}{dz}\left(\dfrac{dy}{dz}\right) = \dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right)\cdot\dfrac{dx}{dz} = x\left\{x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\right\} = x^2D^2y + xDy$

i.e., $D'^2y = x^2D^2y + D'y$ i.e., $x^2D^2y = D'^2y - D'y$ ...(3)

Using (2) and (3) in (1), we have,

$$D'^2y - D'y - D'y + y = z \text{ i.e., } (D'^2 - 2D' + 1)y = z \quad \text{...(4)}$$

To solve (4), first we will consider the reduced equation as

$$(D'^2 - 2D' + 1)y = 0 \quad \text{...(5)}$$

Considering $y = ce^{mz}$ $(c \neq 0)$, the A.E. becomes

$$m^2 - 2m + 1 = 0, \text{ on solving } m = 1, 1$$

Thus the C.F. of (4) is $y = (A + Bz)e^z$, where A and B are two orbitrary constants.

Next to find P.I. of (4), we have from (4)

$$y_P = \frac{1}{D'^2 - 2D' + 1}(z) = (1 + D'^2 - 2D')^{-1}(z) = [1 - (D'^2 - 2D') + \ldots](z) = z + 2.$$

Thus the general solution of the differential equation (4) is $y = (A + Bz)e^z + z + 2.$

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Substituting $z = \log x$, we have,

The general solution of the given differential equation is $y = (A + B \log x)x + \log ...$

where A and B are two arbitrary constants.

3. Solve : $x^2 \dfrac{d^2y}{dx^2} - 6y = x \log x$ [CP 20...]

Solution Here the given differential equation is $x^2 \dfrac{d^2y}{dx^2} - 6y = x \log x$ ... (1)

To solve the differential equation (1), we substitute, $z = \log x$

so that $\dfrac{dz}{dx} = \dfrac{1}{x}$ giving $\dfrac{dx}{dz} = x$.

Now, $\dfrac{dy}{dz} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dz} = \left(\dfrac{dy}{dx}\right)x = x\dfrac{dy}{dx}$

Again, $\dfrac{d^2y}{dz^2} = \dfrac{d}{dz}\left(\dfrac{dy}{dz}\right) = \dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right)\dfrac{dx}{dz} = x\left\{x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\right\} = x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$

i.e., $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - x\dfrac{dy}{dx} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$ ... (2)

Using (2) and (3) in (1) and introducing the differential operator $D' \equiv \dfrac{d}{dz}$, we have

$D'^2y - D'y - 6y = ze^z$ i.e., $(D'^2 - D' - 6)y = ze^z$ ... (3)

To find the C.F. of (3), the reduced equation of (3) is

$(D'^2 - D' - 6)y = 0$ ... (4)

Let $y = ce^{mz}$ $(c \neq 0)$ be a trial solution of (4).

Then the A.E. becomes as $m^2 - m - 6 = 0$ solving $m = 3, -2$.

Therefore, the C.F. of the equation (3) is $y_C = Ae^{3z} + Be^{-2z}$

To find the P.I. of (3), we have from (3)

$$y_P = \frac{1}{D'^2 - D' - 6}(ze^z) = e^z\frac{1}{(D'+1)^2 - (D'+1) - 6}(z) = e^z\frac{1}{D'^2 + D' - 6}(z)$$

$$= \frac{e^z}{-6}\left(1 - \frac{D' + D'^2}{6}\right)^{-1}(z) = -\frac{e^z}{6}\left(1 + \frac{D' + D'^2}{6} + \dots\right)(z) = -\left(z + \frac{1}{6}\right)\frac{e^z}{6} = -\frac{e^z(6z+1)}{36}.$$

Thus the general solution of (3) is $y = y_c + y_p$

i.e., $y = Ae^{3z} + Be^{-2z} - \dfrac{(6z+1)e^z}{36}$.

Substituting $z = \log x$, we have the required general solution of the given equation (1) is

$$y = Ax^3 + \frac{B}{x^2} - \frac{(6\log x + 1)x}{36}$$

4. Solve : $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 2y = \cos(\log x)$ [CP 1999, 2015]

Solution Here the given differential equation is

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = \cos(\log x) \quad ...(1)$$

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To solve (1), let us substitute $\log x = z$ so that we have $x = e^z$ and $\frac{dx}{dz} = e^z = x$.

Now, $\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = x\frac{dy}{dx}$ i.e., $x\frac{dy}{dx} = \frac{dy}{dz}$ ...(2)

Also, $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\cdot\frac{dx}{dz} = \left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)x$

i.e., $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - x\frac{dy}{dx} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ ...(3)

Using (2) and (3) is (1), the differential equation (1) in reduced to

$$\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right) + 3\frac{dy}{dz} + 2y = \cos z \quad i.e., \quad \frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 2y = \cos z$$

i.e., $(D'^2 + 2D' + 2)y = \cos z$, where $D' \equiv \frac{d}{dz}$ ...(4)

The reduced form of the differential equation (4) is $(D'^2 + 2D' + 2)y = 0$ ...(5)

Let $y = ce^{mz}$ $(m \neq 0)$ be a trial solution of (5).

Then the A.E. becomes $m^2 + 2m + 2 = 0$ and on solving, we have

$$m = \frac{1}{2}\left(-2\pm\sqrt{4-8}\right) = -1 \pm i.$$

Therefore the C.F. of (4) is $y_c = e^z(A\cos z + B\sin z)$

where A, B are two arbitrary constants.

To find P.I., we have from (4),

$$y_P = \frac{1}{D'^2+2D'+2}(\cos z) = \frac{D'^2-2D'+2}{(D'^2+2D'+2)(D'^2-2D'+2)}(\cos z)$$

$$= (D'^2 - 2D' + 2)\frac{1}{(D'^2+2)^2-4D'^2}(\cos z)$$

$$= (D'^2 - 2D' + 2)\frac{1}{(-1+2)^2-4(-1)}(\cos z) = \frac{1}{5}(D'^2 - 2D' + 2)(\cos z)$$

$$= \frac{1}{5}(-\cos z + 2\sin z + 2\cos z) = \frac{1}{5}(\cos z + 2\sin z)$$

Thus the required general solution of (4) is $y = y_c + y_p$

i.e., $y = e^z(A\cos z + B\sin z) + \frac{1}{5}(\cos z + 2\sin z)$

Putting $z = \log x$ above the general solution of given equation is

$y = x\{A\cos(\log x) + B\sin(\log x)\} + \frac{1}{5}\{\cos(\log x) + 2\sin(\log x)\}$, where A and B are two constants.

■ Alternative Method for finding P.I.

$$y_P = \frac{1}{D'^2+2D'+2}(\cos z) = \frac{1}{-1^2+2D'+2}(\cos z) = \frac{1}{2D'+1}(\cos z) = \frac{2D'-1}{4D'^2-1}(\cos z)$$

$$= \frac{-2\sin z - \cos z}{4(-1^2)-1} = \frac{1}{5}(2\sin z + \cos z)$$

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5. Solve : $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=\log x^2$

Solution Here the given differential equation is

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=\log x^2$$

To solve (1), let us substitute $z = \log x$ *i.e.*, $x = e^z$, so that $\frac{dx}{dz} = e^z = x$.

Now, $\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = x\frac{dy}{dx}$ *i.e.*, $x\frac{dy}{dx}=\frac{dy}{dz}$

Also, $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dz} = x\left\{x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right\} = x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}$

i.e., $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}-x\frac{dy}{dx} = \frac{d^2y}{dz^2}-\frac{dy}{dz}$

Substituting (2) and (3) in (1), we get

$$\frac{d^2y}{dz^2}-\frac{dy}{dz}-4\frac{dy}{dz}+6y = 2z \quad i.e., \ (D'^2-5D'+6)y = 2z, \text{ where } D'\equiv\frac{d}{dz}$$

To solve (4), the reduced equation of (4) can be written as

$$(D'^2-5D'+6)y = 0$$

Let $y = ce^{mz}$ $(c \neq 0)$ be a trial solution of (5).

Then the A.E. is $m^2 - 5m + 6 = 0$ on solving we have $m = 2, 3$.

Therefore the C.F. of (4) is $y_c = Ae^{2z} + Be^{3z}$, where A, B are two arbitrary constants

To find the P.I. of (4), we have

$$y_P = \frac{1}{D'^2-5D'+6}(2z) = \frac{2}{6}\left(1+\frac{D'^2-5D'}{6}\right)^{-1}(z)$$

$$= \frac{1}{3}\left(1-\frac{D'^2-5D'}{6}+\ldots\right)(z) = \frac{1}{3}\left(z+\frac{5}{6}\right) = \frac{1}{18}(5+6z)$$

Therefore the general solution of the system (4) is

$$y = Ae^{2z}+3e^{3z}+\frac{1}{18}(5+6z)$$

Now substituting $z = \log x$, the general solution of the given differential equation can be written as $y = Ax^2 + Bx^3 + \frac{1}{18}(5 + 6\log x)$, where A and B are two arbitrary constants.

6. Solve : $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=6x^5$

[CP 1996, 2000, 2009]

Solution Here the given differential equation is $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=6x^5$ ... (1)

To solve (1), let us substitute $x = e^z$ (*i.e.*, $z = \log x$)

so that $\frac{dx}{dz} = e^z = x$

Now, $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$ i.e., $\frac{dy}{dz} = x\frac{dy}{dx}$ ... (2)

Also, $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dz} = x\left\{x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right\} = x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}$

i.e., $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - x\frac{dy}{dx} = \frac{d^2y}{dz^2}-\frac{dy}{dz}$ ...(3)

Using (2) and (3) in (1), we get $\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right) - 4\frac{dy}{dz}+6y = 6e^{5z}$

i.e., $\frac{d^2y}{dz^2} - 5\frac{dy}{dz}+6y = 6e^{5z}$ ...(4)

To solve (4), the reduced equation of (4) becomes

$\frac{d^2y}{dz^2} - 5\frac{dy}{dz}+6y = 0$ ...(5)

Let $y = ce^{mz}$ ($c \neq 0$) be a trial solution of (5)

Then the A.E. becomes $m^2 - 5m + 6 = 0$ and on solving we have $m = 2, 3$.

Therefore the C.F. of the equation (4) is

$y_C = Ae^{2z} + Be^{3z}$, where A, B are two orbitrary constants.

To find P.I. of (4), we have from (4),

$y_P = \frac{1}{D'^2-5D'+6}(6e^{5z})$, $D' \equiv \frac{d}{dz}$

$= 6\cdot\frac{1}{D'^2-5D'+6}(e^{5z}) = 6\cdot\frac{1}{(5)^2-5\times5+6}e^{5z} = e^{5z}$.

Therefore, the general solution of the equation (4) is

$y = y_C + y_P = Ae^{2z} + Be^{3z} + e^{5z}$.

Substituting $x = e^z$, we have the general solution of the given differential equation as

$y = Ax^2 + Bx^3 + x^5$, where A, B are two arbitrary constants.

7. Solve : $(x+1)^2\frac{d^2y}{dx^2} - 4(x+1)\frac{dy}{dx}+4y = x^2$ [CP 2012]

Solution Here the given equation is $(x+1)^2\frac{d^2y}{dx^2} - 4(x+1)\frac{dy}{dx}+4y = x^2$ ...(1)

To solve (1), we substitute, $x + 1 = e^z$ [*i.e.*, $z = \log(x+1)$]

so that $\frac{dx}{dz} = e^z = x + 1$.

Now, $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = (x+1)\frac{dy}{dx}$ ...(2)

and $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left\{(x+1)\frac{dy}{dx}\right\}\frac{dx}{dz} = (x+1)\left\{(x+1)\frac{d^2y}{dx^2}+\frac{dy}{dx}\right\}$

$= (x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx}$

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i.e., $(x+1)^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - (x+1)\dfrac{dy}{dx} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$

Using (2) and (3) in (1), we have $\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz} - 4\dfrac{dy}{dz} + 4y = (e^z - 1)^2$

i.e., $\dfrac{d^2y}{dz^2} - 5\dfrac{dy}{dz} + 4y = e^{2z} - 2e^z + 1$

To solve (4), the reduced equation of (4) becomes, $\dfrac{d^2y}{dz^2} - 5\dfrac{dy}{dz} + 4y = 0$

Let $y = ce^{mz}$, $(c \neq 0)$ be a trial solution of (5), then the A.E. becomes $m^2 - 5m + 4 =$ solving we have $m = 1, 4$.

Therefore the C.F. of (4) is $y_C = Ae^z + Be^{4z}$, where A and B are two arbitrary constants.

To find P.I. of (4), we have from (4),

$$y_P = \frac{1}{(D'^2 - 5D' + 4)}(e^{2z} - 2e^z + 1),\ D' \equiv \frac{d}{dz}$$

$$= \frac{1}{D'^2 - 5D' + 4}(e^{2z}) - 2\frac{1}{D'^2 - 5D' + 4}(e^z) + \frac{1}{D'^2 - 5D' + 4}(1)$$

$$= e^{2z}\frac{1}{4 - 10 + 4} - 2e^z\frac{1}{(D'+1)^2 - 5(D'+1) + 4}(1) + \frac{1}{4}\left(1 + \frac{D'^2 - 5D'}{4}\right)^{-1}(1)$$

$$= -\frac{1}{2}e^{2z} - 2e^z\frac{1}{D'^2 - 3D'}(1) + \frac{1}{4}\left(1 - \frac{D'^2 - 5D'}{4} + \ldots\right)(1)$$

$$= -\frac{1}{2}e^{2z} - 2e^z\frac{1}{-3D'}\left(1 - \frac{D'}{3}\right)^{-1}(1) + \frac{1}{4} = \frac{1}{4} - \frac{1}{2}e^{2z} + \frac{2}{3}e^z\frac{1}{D'}(1) = \frac{1}{4} - \frac{1}{2}e^{2z} + \frac{2z}{3}e^z.$$

Therefore the general solution of (4) is $y = y_C + y_P$

i.e., $y = Ae^z + Be^{4z} + \dfrac{1}{4} - \dfrac{1}{2}e^{2z} + \dfrac{2z}{3}e^z$.

Substituting $z = \log(x+1)$, the general solution of the given differential equation is

$$y = A(x+1) + B(x+1)^4 + \frac{1}{4} - \frac{1}{2}(x+1)^2 + \frac{2}{3}(x+1)\log(x+1)$$

$$= A_1x + B(x+1)^4 - \frac{1}{2}(x+1)^2 + \frac{2}{3}(x+1)\log(x+1)$$

where $A_1 = A + \dfrac{1}{4}$ and B are two arbitrary constants.

8. **Solve :** $(x+a)^2\dfrac{d^2y}{dx^2} - 5(x+a)\dfrac{dy}{dx} + 6y = 0$

[CP 2013]

Solution Here the given differential equation is

$$(x+a)^2\frac{d^2y}{dx^2} - 5(x+a)\frac{dy}{dx} + 6y = 0 \quad \ldots(1)$$

To solve (1), let us substitute $(x+a) = e^z$ (*i.e.*, $z = \log(x+a)$) so that we have

$\dfrac{dx}{dz} = e^z = x + a$.

Now, $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = (x+a)\frac{dy}{dx}$ ...(2)

Also, $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left((x+a)\frac{dy}{dx}\right)\frac{dx}{dz}$

$$= (x+a)\left\{(x+a)\frac{d^2y}{dx^2}+\frac{dy}{dx}\right\} = (x+a)^2\frac{d^2y}{dx^2}+(x+a)\frac{dy}{dx}$$

i.e., $(x+a)^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}-(x+a)\frac{dy}{dx} = \frac{d^2y}{dz^2}-\frac{dy}{dz}$ ...(3)

Using (2) and (3) in (1), we have, $\frac{d^2y}{dz^2}-\frac{dy}{dz}-5\frac{dy}{dz}+6y=0$

i.e., $\frac{d^2y}{dz^2}-6\frac{dy}{dz}+6y=0$ ...(4)

Let $y = ce^{mz}$ $(c \neq 0)$ be a trial solution of (4). Then the A.E. of (4) is $m^2-6m+6=0$

on solving $m = \frac{1}{2}\left(6\pm\sqrt{36-24}\right) = \frac{1}{2}\left(6\pm2\sqrt{3}\right) = 3\pm\sqrt{3}$.

Therefore the general solution of (4) will be $y = Ae^{(3+\sqrt{3})z}+Be^{(3-\sqrt{3})z}$,

A and B are arbitrary constants.

Substituting, $e^z = (x+a)$, the general solution of the given system can be written as

$y = A(x+a)^{3+\sqrt{3}}+B(x-a)^{3-\sqrt{3}}$, where A and B are two arbitrary constants.

9. Solve : $(1+2x)^2\frac{d^2y}{dx^2}-8(1+2x)\frac{dy}{dx}+16y=0$ [CP 2014]

Solution Here the given differential equation is

$$(1+2x)^2\frac{d^2y}{dx^2}-8(1+2x)\frac{dy}{dx}+16y=0 \quad ...(1)$$

To solve (1), let us substitute, $(1+2x) = e^z$ (i.e., $z = \log(1+2x)$)

so that we have, $2\frac{dx}{dz} = e^z = (1+2x)$ i.e., $\frac{dx}{dz} = \frac{1+2x}{2}$

Now, $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{1+2x}{2}\cdot\frac{dy}{dx}$ i.e., $(1+2x)\frac{dy}{dx} = 2\frac{dy}{dz}$ ...(2)

Again, $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left\{\left(\frac{1+2x}{2}\right)\frac{dy}{dx}\right\}\frac{dx}{dz} = \frac{1+2x}{2}\left\{\frac{(1+2x)}{2}\frac{d^2y}{dx^2}+\frac{dy}{dx}\right\}$

$$= \frac{(1+2x)^2}{4}\frac{d^2y}{dx^2}+\frac{(1+2x)}{2}\frac{dy}{dx}$$

i.e., $(1+2x)^2\frac{d^2y}{dx^2} = 4\frac{d^2y}{dz^2}-2(1+2x)\frac{dy}{dx} = 4\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$ ...(3)

Substituting results from (2) and (3) in (1), we have

$$4\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)-8\times2\frac{dy}{dz}+16y=0$$

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i.e., $4\frac{d^2y}{dz^2} - 20\frac{dy}{dz} + 16y = 0$ i.e., $\frac{d^2y}{dz^2} - 5\frac{dy}{dz} + 4y = 0$ ...(4)

Let $y = ce^{mz}$ $(c \neq 0)$ be a trial solution of (4).

Then the A.E. of (4) is $m^2 - 5m + 4 = 0$, on solving we have $m = 1, 4$.

Therefore the general solution of (4) is $y = Ae^z + Be^{4z}$; A, B are arbitrary constants.

Putting $e^z = (1 + 2x)$, the general solution of given equation is

$$y = A(1 + 2x) + B(1 + 2x)^4;\ \text{where A, B are arbitrary constants.}$$

10. **Solve : $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2$; given that $y = 0$ when $x = 1$ and $y = e^2$ when $x = e$.** [BP 1989]

Solution Here the given differential equation is $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2$ ...(1)

To solve (1), we substitute $x = e^z$ (i.e. $z = \log x$) so that we have $\frac{dx}{dz} = e^z = x$.

Now, $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz}$ i.e., $x\frac{dy}{dx} = \frac{dy}{dz}$ ...(2)

Also, $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dz} = x\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$

i.e., $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - x\frac{dy}{dx} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ ...(3)

Using (2) and (3) in (1), the differential equation (1) reduces to

$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 3\frac{dy}{dz} + 4y = e^{2z}$ i.e., $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = e^{2z}$

i.e., $(D'^2 - 4D' + 4)y = e^{2z}$; $D' = \frac{d}{dz}$ ...(4)

To solve (4), the corresponding reduced equation is as follows :

$(D'^2 - 4D' + 4)y = 0$ ...(5)

Let $y = ce^{mz}$ $(c \neq 0)$ be a trial solution of (5). Then the A.E. of (5) is

$m^2 - 4m + 4 = 0$, on solving we have $m = 2, 2$.

Thus the C.F. of (4) is $y_C = (A + Bz)e^{2z}$, where A, B are two arbitrary constants.

To find P.I. of (4), we have from (4),

$$y_P = \frac{1}{D'^2 - 4D' + 4}(e^{2z}) = \frac{1}{(D' - 2)^2}(e^{2z}) = e^{2z}\frac{1}{(D' + 2 - 2)^2}(1) = e^{2z}\frac{1}{D'^2}(1) = \frac{z^2e^{2z}}{2}$$

Therefore the general solution of the equation (4) is

$$y = y_C + y_P = \left(A + Bz + \frac{z^2}{2}\right)e^{2z}$$

Substituting $z = \log x$, we have the general solution of the given differential equation as

$$y = \left(A + B\log x + \frac{(\log x)^2}{2}\right)x^2;\ \text{where A, B are two arbitrary constants.}$$

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Using the boundary conditions, $y = 0$ when $x = 1$, we have,

$$0 = \left(A + B\log 1 + \frac{(\log 1)^2}{2}\right)1 \quad i.e.,\ A = 0$$

Again using $y = e^2$, when $x = e$, we have,

$$e^2 = \left(A + B\log e + \frac{(\log e)^2}{2}\right)e^2$$

i.e., $B + \frac{1}{2} = 1$, since $A = 0$, *i.e.*, $B = \frac{1}{2}$.

Therefore substituting $A = 0, B = \frac{1}{2}$, the general solution of the given differential equation which satisfies the given boundary conditions is

$$y = \left(\frac{1}{2}\log x + \frac{1}{2}(\log x)^2\right)x^2 = \frac{x^2}{2}\{\log x + (\log x)^2\}$$

[VH 1997]

11. **Solve : $(x^3D^3 + xD - 1)y = x^2$, $D \equiv \frac{d}{dx}$**

Solution Here the given differential equation is $(x^3D^3 + xD - 1)y = x^2$ ... (1)

To solve (1), we put $\log x = z$, so that we have, $x = e^z$ and $\frac{dx}{dz} = e^z = x$.

Now, $\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = x\frac{dy}{dx}$ *i.e.*, $x\frac{dy}{dx} = \frac{dy}{dz}$ *i.e.*, $xDy = D'y$, where $D' \equiv \frac{d}{dz}$

Also, $\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dz} = x\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right)$

i.e., $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ *i.e.*, $x^2D^2y = (D'^2 - D')y = D'(D' - 1)y$

Similarly, $x^3D^3y = D'(D' - 1)(D' - 2)y$

Using the above transformations, (1) reduces to :

$D'(D' - 1)(D' - 2)y + D'y - y = e^{2z}$ *i.e.*, $(D'^3 - 3D'^2 + 3D' - 1)y = e^{2z}$ ... (2)

i.e., $(D' - 1)^3y = e^{2z}$

To solve (2), the reduced equation becomes : ... (3)

$$(D' - 1)^3y = 0$$

Let $y = ce^{mz}$ be a trial solution of (3). Then the auxiliary equation of (3) is

$$(m - 1)^3 = 0, \text{ giving } m = 1, 1, 1.$$

Thus the C.F. of (2) is $y_C = Ae^z + Bze^z + Cz^2e^z$, where A, B, C are three arbitary constants.

If y_P be the P.I. of (2), then from (2) we have,

$$y_P = \frac{1}{(D'-1)^3}e^{2z} = \frac{1}{(2-1)^3}e^{2z} = e^{2z}$$

Thus the general solution of the given differential equation is $y = y_C + y_P$

i.e., $y = Ae^z + Bze^z + Cz^2e^z + e^{2z}$

i.e., $y = Ax + Bx\log x + Cx(\log x)^2 + x^2$, where A, B, C are three arbitrary constants.

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12. Solve : $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ [CH 1991, 1993, ...

Solution Here the given differential equation is $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

To solve (1) we substitute $\log x = z$ *i.e.*, $e^z = x$.

Now $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = x\frac{dy}{dx}$ *i.e.*, $xDy = D'y$, where $D \equiv \frac{d}{dx}$, $D' \equiv \frac{d}{dz}$

Again, $\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dz} = x\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + \frac{dy}{dz}$

i.e., $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dx^2} - \frac{dy}{dz} = (D'^2 - D')y = D'(D' - 1)y$

Similarly we can write, $x^2\frac{d^3y}{dx^3} = D'(D' - 1)(D' - 2)y$

Substituting all of these in (1) we have,

$\{D'(D' - 1)(D' - 2) + 2D'(D' - 1) + 2\}y = 10(e^z + e^{-z})$

i.e., $(D'^3 - D'^2 + 2)y = 10(e^z + e^{-z})$ –(2)

The reduced equation of (2) is $(D'^3 - D'^2 + 2)y = 0$ –(3)

Let, $y = ce^{mz}$ be a trial solution of (3). Then the auxiliary equation of (3) becomes

$m^3 - m^2 + 2 = 0$

i.e., $(m + 1)(m^2 - 2m + 2) = 0$ giving $m = -1, 1 \pm i$.

Thus the complementary function of (3) is given by

$y_C = Ae^{-z} + e^z(B\cos z + C\sin z)$, where A, B, C are arbitrary constants.

To find Particular integral y_P, we have from (2),

$$y_P = \frac{1}{D'^3 - D'^2 + 2}\{10(e^z + e^{-z})\} = 10\frac{1}{(D'+1)(D'^2 - 2D' + 2)}(e^z + e^{-z})$$

$$= 10\left[\frac{1}{(D'+1)(D'^2 - 2D' + 2)}(e^z) + \frac{1}{(D'+1)(D'^2 - 2D' + 2)}(e^{-z})\right]$$

$$= 10\left[\frac{1}{(1+1)(1-2+2)}e^z + e^{-z}\frac{1}{(D'-1+1)\{(D'-1)^2 - 2(D'-1) + 2\}}(1)\right]$$

$$= 10\left[\frac{1}{2}e^z + e^{-z}\frac{1}{D'(D'^2 - 4D' + 5)}(1)\right] = 5e^z + \frac{10}{5}e^{-z}\frac{1}{D'}\left(1 + \frac{D'^2 - 4D'}{5}\right)^{-1}(1)$$

$$= 5e^z + 2e^{-z}\frac{1}{D'}\left(1 - \frac{D'^2 - 4D'}{5} + \ldots\right)(1)$$

$$= 5e^z + 2e^{-z}\frac{1}{D'}(1) = 5e^z + 2ze^{-z}$$

Therefore the general solution is $y = y_C + y_P = Ae^{-z} + e^z(B\cos z + C\sin z) + 5e^z + 2ze^{-z}$

i.e., $y = (A + 2z)e^{-z} + (B\cos z + C\sin z + 5)e^z = \frac{A + 2\log x}{x} + x\{B\cos(\log x) + C\sin(\log x) + 5\}$,

where A, B, C are three arbitrary constants.

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Exercise

- **Solve the following differential equations :**

1. $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+2y=\cos(\log x)$ **[CP 1999]**

2. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-4y=x$ **[CP 1987]**

3. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+4y=0$ **[CP 1993]**

4. $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=\log x^2$ **[BH 2001]**

5. $x^2y''-2xy'+2y=x^4e^x$ **[BH 2009, '06]**

6. $(x+a)^2\dfrac{d^2y}{dx^2}-4(x+a)\dfrac{dy}{dx}+6y=x$ **[CH 1995, 2002]**

7. $(2+x)^2\dfrac{d^2y}{dx^2}+(2+x)\dfrac{dy}{dx}+4y$
$=2\sin\{2\log(2+x)\}$ **[CH 2006]**

8. $x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-\dfrac{3}{x}y=\dfrac{x^2\log x+5}{x}$ **[CH 2008]**

9. $(3x+2)^2y''+5(3x+2)y'-3y$
$=x^2+x+1$

10. $x^2y''-xy'+2y=\sin(\log x)$ **[CP 1990]**

11. $x^2y''+3xy'+y=x\log x$ **[CP 1992]**

12. $x^2\dfrac{d^2y}{dx^2}+y=3x^2$ **[CP 1988, BP 1999]**

13. $x^2y''-xy'+y=2\log x$ **[CP 1991, 1994]**

14. $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+4y=2y^3$ **[CP 1997]**

15. $(x^3D^3+3x^2D^2+xD+1)y=x\log x,$
$D\equiv\dfrac{d}{dx}$

Answers

1. $y=\dfrac{1}{x}\{A\cos(\log x)+B\sin(\log x)\}+$
$\dfrac{1}{5}\{2\sin(\log x)+\cos(\log x)\}$

2. $y=Ax^2+\dfrac{B}{x^2}-\dfrac{1}{3}x$

3. $y=A\cos(2\log x)+B\sin(2\log x)$

4. $y=Ax^2+Bx^3+\dfrac{1}{18}(6\log x+5)$

5. $y=Ax+Bx^2+x(x-2)e^x$

6. $y=A(x+a)^2+B(x+a)^3+\dfrac{1}{2}(x+a)-\dfrac{1}{6}a$

7. $y=A\cos\{2\log(2+x)\}+B\sin\{2\log(2+x)\}$
$-\log(2+x)\cos\{2\log(2+x)\}$

8. $y=\dfrac{A_1}{x}+Bx^3-\dfrac{x^2}{9}(3\log x+2)-\dfrac{5}{3}$

9. $y=A(3x+2)^{-1}+B(3x+2)^{\frac{1}{3}}-\dfrac{7}{27}-$
$\dfrac{1}{108}(3x+2)+\dfrac{1}{405}(3x+2)^2$

10. $y=x\{A\cos(\log x)+B\sin(\log x)\}+$
$\dfrac{1}{5}\{2\cos(\log x)+\sin(\log x)\}$

11. $y=\dfrac{1}{2}(A+B\log x)+\dfrac{1}{4}x(\log x-1)$

12. $y=\{A\cos(\sqrt{3}/2\log x)+$
$B\sin\left(\sqrt{3}/2\log x\right)\sqrt{x}+x^2\}$

13. $y=x(A+B\log x)+4+2\log x$

14. $y=(A+B\log x)x^2+2x^3$

15. $y=\dfrac{A}{x}+\sqrt{x}\left\{B\cos\dfrac{\sqrt{3}\log x}{2}+C\sin\dfrac{\sqrt{3}\log x}{2}\right\}$
$+\dfrac{x}{4}(2\log x-3)$

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9

CHAPTER

Method of Undetermined Coefficients

Introduction

Let us consider a second order differential equation as follows :

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2y = \phi(x) \quad \text{...(1)}$$

The differential equation stated in (1) can be solved by the method of undetermined coefficients if

1. p_1 and p_1 are constants only,
2. $\phi(x)$ is a function of x such that it may be either of exponential function, sine function, cosine function, polynomial function or product of these types of functions.

In any other cases except the cases (1) and (2), the solution of the differential equation (1) cannot be found by method of undetermined coefficients.

Consulting from earlier chapter, it is to be concluded that the general solution of the differential equation (1) contains two parts, one is complementary function (C.F.) and another is particular integral (P.I.). If y_C and y_P be the complementary function and particular integral of (1), then the general solution of (1) will be $y = y_C + y_P$ ...(2)

The complementary function y_C can be found by solving the auxiliary equation (A.E.) which can be derived from the corresponding reduced equation of (1) is $\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2y = 0$. [Details are already discussed in earlier chapter].

However to determine the P.I., y_P we use the method of undetermined coefficient. Below we state the rules to determine y_P for various types of function of $\phi(x)$.

Rule-1 : Let $\phi(x)$ is an exponential function, *i.e.*, let $\phi(x) = ke^{ax}$ where k and a both are constants.

(i) If a is not a root of the A.E. then take $y_P = ce^{ax}$, c is a constant to be determined.

(ii) If a is a simple root of A.E., then take $y_P = cxe^{ax}$, c is a constant to be determined.

(iii) If a is a double root of A.E., then take $y_P = cx^2e^{ax}$, c is constant to be determined.

Rule-2 : If $\phi(x)$ is a sine function or cosine function or a linear combination of sine and cosine function *i.e.*, if $\phi(x) = k_1\sin\alpha x + k_2\cos\alpha x$, where k_1, k_2 and α are constants.

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(i) If both of $\sin\alpha x$ and $\cos\alpha x$ is absent in the C.F., then choose $y_P = c_1\sin\alpha x + c_2\cos\alpha x$; c_1, c_2 are constants to be determined.

(ii) If both or either of $\sin\alpha x$ and $\cos\alpha x$ remains in the C.F., then choose $y_P = x(c_1\sin\alpha x + c_2\cos\alpha x)$; c_1, c_2 are constants to be determined.

Rule-3 : If $\phi(x)$ is a n-th order polynomial function of x, *i.e.*, say,
$\phi(x) = k_0 + k_1x + k_2x^2 + ... + k_nx^n$; where $k_0, k_1, ..., k_n$ are constants.

(i) If $p_1 \neq 0$ and $p_2 \neq 0$, then take $y_P = c_0 + c_1x + c_2x^2 + ... + c_nx^n$, where $c_0, c_1, ..., c_n$ are constants to be determined.

(ii) If $p_1 \neq 0$ but $p_2 = 0$, then assume $y_P = x(c_0 + c_1x + c_2x^2 + ... + c_nx^n)$, where $c_0, c_1, ..., c_n$ are constants to be determined.

(iii) If both $p_1 = 0$ and $p_2 = 0$, then assume $y_P = x^2(c_0 + c_1x + c_2x^2 + ... + c_nx^n)$, where $c_0, c_1, ..., c_n$ are constants to be determined.

Notes

1. If both $p_1, p_2 = 0$, then direct integration gives result more quickely.
2. If $\phi(x) = k$ = constant then choose $y_P = c$ (constant)
3. If $\phi(x)$ is a product of the functions discussed in Rule-1, Rule-2 and Rule-3, then y_P is also taken as the product form of corresponding particular integral function. As for an example if $\phi(x) = xe^x$ and both of x and e^x is absent in y_C then choose $y_P = (c_0 + c_1x)e^x$ where c_0, c_1 are constants to be determined.

Procedure to determine the constant(s) : Clearly the chosen P.I., y_P contains some constant(s) which can be determined by substituting y_P in the differential equation

$$D^2y_P + a_1Dy_P + a_2 = \phi(x) \quad [\text{where } D \equiv \frac{d}{dx}] \qquad .... (3)$$

Illustrative Examples

1. **Solve by method of undetermined coefficients the following equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 8e^x$.**

Solution Introducing $D \equiv \frac{d}{dx}$, the given differential equation can be written as

$$(D^2 - 5D + 6)y = 8e^x \qquad ... (1)$$

The reduced form of the equation (1) is $(D^2 - 5D + 6)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2).

Then the A.E. becomes as $m^2 - 5m + 6 = 0$, on solving we have $m = 2, 3$

Since the two roots of the A.E. are real and unequal, therefore the C.F. of the given equation is $y_C = Ae^{2x} + Be^{3x}$ where A, B are two arbitrary constants.

Since the function e^x is absent in the expression of C.F., y_C, then let us take the P.I.,

$y_P = Ce^x$ where C is a constant to be determined from the following equation, ... from (1),

$(D^2 - 5D + 6)y_P = 8e^x$ i.e., $(D^2 - 5D + 6)ce^x = 8e^x$

i.e., $(c - 5c + 6c) = 8$, since $e^x \neq 0$ i.e., $2c = 8$ giving $c = 4$ Thus $y_P = 4e^x$.

Hence the required solution is $y = y_C + y_P = Ae^{2x} + Be^{3x} + 4e^x$, where A, B ... arbitrary constants.

2. **Solve by method of undetermined coefficients, the following equ...**

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6e^{2x}$$

Solution Introducing the differential equation $D \equiv \frac{d}{dx}$, the given equation can be writte...

$$(D^2 - 5D + 6)y = 6e^{2x} \quad \text{... (1)}$$

Proceeding as the same way discussed in illustrative example-1, the C.F. of the probl... (1) is $y_C = Ae^{2x} + Be^{3x}$, where A and B are two arbitrary constants.

Since the function e^{2x} is within the C.F. once, then let us choose the P.I., $y_P = cx...$ where c is a constant to be determined from the equation.

$(D^2 - 5D + 6)y_P = 6e^{2x}$ i.e., $(D^2 - 5D + 6)(cxe^{2x}) = 6e^{2x}$

i.e., $c[D^2(xe^{2x}) - 5D(xe^{2x}) + 6xe^{2x}) = 6e^{2x}$

i.e., $c[4(xe^{2x} + e^{2x}) - 5(2xe^{2x} + e^{2x}) + 6xe^{2x}] = 6e^{2x}$

i.e., $-ce^{2x} = 6e^{2x}$ giving $c = -6$, since $e^{2x} \neq 0$

Therefore we have, $y_P = -6xe^{2x}$.

Hence the required general solution is $y = (A - 6x)e^{2x} + Be^{3x}$, where A and B are tw... arbitrary constants.

3. **Solve by the method of undetermined coefficients $(D^2 + 6D + 9)y = 24e^{-3x}$, where $D \equiv \frac{d}{dx}$.**

[CH 1999]

Solution Here the given differential equation is $(D^2 + 6D + 9)y = 24e^{-3x}$ ... (1)

To solve (1), the corresponding reduced equation of (1) is

$(D^2 + 6D + 9)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of the equation (2)

Then the A.E. becomes $m^2 + 6m + 9 = 0$, on solving $m = -3, -3$.

Since the A.E. has real and equal root, then the C.F. is $y_C = (A + Bx)e^{-3x}$, where A, B are two constants.

Since -3 is a double root of A.E. (i.e., e^{-3x} appears twice) in the C.F. then we choose, P.I. of (1) as $y_P = cx^2e^{-3x}$ where c is constant and can be determined from the equation

$(D^2 + 6D + 9)y_P = 24e^{-3x}$ ... (3)

From (3) we have, $(D^2 + 6D + 9)(cx^2e^{-3x}) = 24e^{-3x}$

i.e., $D^2(cx^2e^{-3x}) + 6D(cx^2e^{-3x}) + 9(cx^2e^{-3x}) = 24e^{-3x}$

i.e., $ce^{-3x}(9x^2 - 12x + 2) + 6ce^{-3x}(-3x^2 + 2x) + 9cx^2e^{-3x} = 24e^{-3x}$

i.e., $2ce^{-3x} = 24e^{-3x}$, on solving $c = 12$.

Therefore P.I. is $y_P = 12x^2e^{-3x}$

Combining y_C and y_P, the general solution of the given differential equation is $y = y_C + y_P = (A + Bx + 12x^2)e^{-3x}$ where A, B are two arbitrary constants.

4. **Solve by method of undetermined coefficients the equation $y'' - 3y' + 2y = \sin 2x$** **[CH 2003]**

Solution Introducing $D \equiv \frac{d}{dx}$, the given differential equation can be written as

$$(D^2 - 3D + 2)y = \sin 2x \quad \dots (1)$$

The reduced equation of (1) is $(D^2 - 3D + 2)y = 0$ ... (2)

Let $y = ce^{mx}$ be a trial solution of (2).

Then the A.E. becomes as $m^2 - 3m + 2 = 0$ on solving we have, $m = 1, 2$.

Since the the roots of the A.E. are real and unequal then the C.F. of (1) will be $y_C = Ae^x + Be^{2x}$, where A, B are two arbitrary constants.

Since $\sin 2x$ does not appear in C.F., then let us take the P.I. as $y_P = c_1\sin 2x + c_2\cos 2x$, where c_1 and c_2 are two constants and can be determined from the equation $(D^2 - 3D + 2)y_P = \sin 2x$ ... (3)

From (3) we have, $(D^2 - 3D + 2)(c_1\sin 2x + c_2\cos 2x) = \sin 2x$

i.e., $D^2(c_1\sin 2x + c_2\cos 2x) - 3D(c_1\sin 2x + c_2\cos 2x) + 2(c_1\sin 2x + c_2\cos 2x) = \sin 2x$

i.e., $(-4c_1\sin 2x - 4c_2\cos 2x) - 6(c_1\cos 2x - c_2\sin 2x) + 2(c_1\sin 2x + c_2\cos 2x) = \sin 2x$

i.e., $(-4c_1 + 6c_2 + 2c_1 - 1)\sin 2x + (-4c_2 - 6c_1 + 2c_2)\cos 2x = 0$

i.e., $(6c_2 - 2c_1 - 1)\sin 2x - (2c_2 + 6c_1)\cos 2x = 0$

To solve, we choose $6c_2 - 2c_1 - 1 = 0$ and $2c_2 + 6c_1 = 0$ giving $c_2 = 3c_1$

i.e., $-20c_1 = 1$ given $c_1 = -\frac{1}{20}$ and $c_2 = \frac{3}{20}$.

Putting the values of c_1 and c_2, we have, $y_P = -\frac{1}{20}\sin 2x + \frac{3}{20}\cos 2x$,

Therefore the general solution of the given equation is

$y = y_C + y_P = Ae^x + Be^{2x} - \frac{1}{20}\sin 2x + \frac{3}{20}\cos 2x$, where A and B are two arbitrary constants.

5. **Solve by the method of undetermined coefficients the differential equation $(D^2 + 4)y = \sin 2x;\ D \equiv \frac{d}{dx}$.** **[1996]**

Solution Here the given differential equation is $(D^2 + 4)y = \sin 2x$ ... (1)

The reduced equation of (1) can be written as $(D^2 + 4)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2).

Then the A.E. becomes $m^2 + 4 = 0$ giving $m = \pm 2i$.

Thus the C.F. of the given differential equation is $y_C = A\cos 2x + B\sin 2x$, A, B are arbitrary constants.

To find P.I. we observe that $\sin 2x$ appears in C.F., so we choose P.I. as $y_P = x(c_1\sin 2x + c_2\cos 2x)$, where c_1, c_2 are constants to be determined from

$$(D^2 + 4)y_P = \sin 2x \quad \dots (3)$$

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From (3), we have $(D^2 + 4)[x(c_1\sin 2x + c_2\cos 2x)] = \sin 2x$

i.e., $D^2[x(c_1\sin 2x + c_2\cos 2x)] + 4x(c_1\sin 2x + c_2\cos 2x) = \sin 2x$

i.e., $-4xc_1\sin 2x - 4xc_2\cos 2x + 4c_1\cos 2x - 4c_2\sin 2x + 4xc_1\sin 2x + 4xc_2\cos 2x = \sin 2x$

i.e., $(4c_2 + 1)\sin 2x - 4c_1\cos 2x = 0$

To solve, let us take $4c_2 + 1 = 0$ and $4c_1 = 0$, giving $c_1 = 0$, $c_2 = -\frac{1}{4}$

Thus the the P.I. of the given equation is $y_P = -\frac{1}{4}x\cos 2x$.

Therefore, the general solution is $y = y_C + y_P$

i.e., $y = A\cos 2x + B\sin 2x - \frac{1}{4}x\cos 2x$, where A, B are two arbitrary constants.

6. **Solve by method of undetermined coefficients $(D^2 - D - 2)y = -2$.**

Solution Here the given differential equation is $(D^2 - D - 2)y = -2$

The reduced equation of (1) is $(D^2 - D - 2)y = 0$ ... (1)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2). ... (2)

Then the A.E. is $m^2 - m - 2 = 0$ and on solving we have $m = -1, 2$

Since the roots of A.E. are real and unequal,

we have the C.F. of (1) is $y_C = Ae^{-x} + Be^{2x}$, where A, B are two arbitrary constants.

To find P.I. by method of undetermined coefficient, we choose $y_P = c$, where c is a constant and can be found from the differential equation $(D^2 - D - 2)y_P = -2$ ... (3)

From (3) we have, $(D^2 - D - 2)c = -2$ i.e., $-2c = -2$ giving $c = 1$.

Thus the P.I. is $y_P = 1$. Hence the general solution of the given differential equation is

$y = y_C + y_P$ i.e., $y = Ae^{-x} + Be^{2x} + 1$, where A, B are two arbitrary constants.

7. **Solve by the method of undetermined coefficients, $(D^2 - 4D + 3)y = 2x^2 + 3x + 4$, $D \equiv \frac{d}{dx}$.**

Solution Here the given differential equation is

$$(D^2 - 4D + 3)y = 2x^2 + 3x + 4,\ D \equiv \frac{d}{dx},$$

The reduced equation of (1) is $(D^2 - 4D + 3)y = 0$ ... (1)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2). ... (2)

Then the A.E. is given by $m^2 - 4m + 3 = 0$ and on solving we have $m = 1, 3$

Since the two roots of the A.E. are real and unequal, therefore the C.F. of the given differential equation is $y_C = Ae^{x} + Be^{3x}$, where A, B are two arbitrary constants.

To find P.I., since the right hand side of the equation (1) is a polynomial of degree 2, we take $y_P = c_1x^2 + c_2x + c_3$, where c_1, c_2, c_3 are constants to be determined from

$$(D^2 - 4D + 3)y_P = 2x^2 + 3x + 4 \quad \text{... (3)}$$

From (3) we have, $(D^2 - 4D + 3)(c_1x^2 + c_2x + c_3) = 2x^2 + 3x + 4$

i.e., $D^2(c_1x^2 + c_2x + c_3) - 4D(c_1x^2 + c_2x + c_3) + 3(c_1x^2 + c_2x + c_3) = 2x^2 + 3x + 4$

i.e., $2c_1 - 8c_1x - 4c_2 + 3c_1x^2 + 3c_2x + 3c_3 = 2x^2 + 3x + 4$

i.e., $3c_1x^2 + (3c_2 - 8c_1)x + 2c_1 + 3c_3 - 4c_2 = 2x^2 + 3x + 4$

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Comparing the coefficients of like powers of x, we have

$3c_1 = 2,\ 3c_2 - 8c_1 = 3$ and $2c_1 + 3c_3 - 4c_2 = 4$

i.e., $c_1 = \frac{2}{3},\ c_2 = \frac{1}{3}\left(3 + \frac{16}{3}\right) = \frac{25}{9},\ c_3 = \frac{1}{3}\left(4 - \frac{4}{3} + \frac{100}{9}\right) = \frac{124}{27}$

Thus the P.I. is $y_P = \frac{2}{3}x^2 + \frac{25}{9}x^2 + \frac{124}{27}$

Hence the general solution of the given differential equation is

$y = y_C + y_P = Ae^x + Be^{3x} + \frac{2}{3}x^2 + \frac{25}{9}x^2 + \frac{124}{27}$, where A, B are two arbitrary constants.

8. **Solve the differential equation $(D^2 - 3D)y = x^2 + 2$ by the method of undetermined coefficients, where $D \equiv \frac{d}{dx}$.**

Solution Here the given differential equation is $(D^2 - 3D)y = x^2 + 2$ ... (1)

The reduced equation of (1) becomes $(D^2 - 3D)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of the equation (2)

Then the A.E. becomes $m^2 - 3m = 0$ and on solving $m = 0, 3$

Since the roots of A.E. are real and unequal, therefore the C.F. of given differential equation is $y_C = Ae^{0x} + Be^{3x}$ i.e., $y_C = A + Be^{3x}$, where A, B are two arbitrary constants.

To determine the P.I., since the right hand side of (1) is a polynomial function of degree 2 and the coefficient of y is zero, then we choose the P.I. as

$y_P = x(c_1x^2 + c_2x + c_3)$, where c_1, c_2, c_3 are three constants to be determined from the ... (3)

equation $(D^2 - 3D)y_P = x^2 + 2$

Therefore from (3), we have $(D^2 - 3D)\{x(c_1x^2 + c_2x + c_3)\} = x^2 + 2$

i.e., $D^2(c_1x^3 + c_2x^2 + c_3x) - 3D(c_1x^3 + c_2x^2 + c_3x) = x^2 + 2$

i.e., $6c_1x + 2c_2 - 9c_1x^2 - 6c_2x - 3c_3 = x^2 + 2$

i.e., $-9c_1x^2 + 6(c_1 - c_2)x + 2c_2 - 3c_3 = x^2 + 2$

Comparing the coefficients of like powers of x, we have

$-9c_1 = 1,\ 6(c_1 - c_2) = 0$ and $2c_2 - 3c_3 = 2$

i.e., $c_1 = -\frac{1}{9},\ c_2 = c_1 = -\frac{1}{9}$ and $c_3 = \frac{1}{3}\left(-\frac{2}{9} - 2\right) = -\frac{20}{27}$

Therefore the P.I. of (1) is given by $y_P = x\left(-\frac{1}{9}x^2 - \frac{1}{9}x - \frac{20}{27}\right)$

Hence the general solution of the given differential equation is

$y = y_C + y_P = A - \frac{20}{27}x - \frac{1}{9}x^2 - \frac{1}{9}x^3 + Be^x$, where A and B are two arbitrary constants.

9. **Solve by method of undetermined coefficients, the following differential equation $\frac{d^2y}{dx^2} = 6x$.**

Solution Here the given differential equation in $\frac{d^2y}{dx^2} = 6x$ ... (1)

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To solve (1) by method of undetermined coefficients, we have

then reduced equation as $\frac{d^2y}{dx^2} = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ ba a trial solution of the differential equation (2), then A.E. is $m^2 = 0$ giving $m = 0, 0$

Thus the C.F. becomes $y_C = (A + Bx)e^{0x} = A + Bx$, where A, B are two arbitrary constants.

To find P.I., we take $y_P = x^2(c_1x + c_2)$ where c_1, c_2 are two constants can be determined from $\frac{d^2y_P}{dx^2} = 6x$

Now $\frac{d^2y_P}{dx^2} = 6x$ gives $\frac{d^2}{dx^2}[x^2(c_1x + c_2)] = 6x$ *i.e.*, $6c_1x + 2c_2 = 6x$.

Equating the like powers of x, we have $6c_1 = 6$, $2c_2 = 0$ giving $c_1 = 1$, $c_2 = 0$.

Thus the P.I. is $y_P = x^2(x + 0) = x^3$.

Hence the general solution of the given differential equation is $y = A + Bx + x^3$, where A, B are two arbitrary constants.

Note

It should be noted that the solution can easily be obtained by twice integration since $y = \int[\int 6x\,dx]dx = \int(3x^2 + B)dx = x^3 + Bx + A$, where A, B are two arbitrary constants.

10. **Solve by method of undetermined coefficient $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 12x^2 + x - 6$, given $y = 6$ and $\frac{dy}{dx} = 11$ when $x = 0$** [CH 2008]

Solution Here the given differential equation is

$$(D^2 - 4D + 3)y = 12x^2 + x - 6,\ D \equiv \frac{d}{dx} \quad ...(1)$$

The reduced equation of (1) becomes as $(D^2 - 4D + 3)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of the differential equation (2)

Then the A.E. becomes $m^2 - 4m + 3 = 0$, solving $m = 1, 3$.

Since both the roots of A.E. are real and unequal therefore, the C.F. of the given equation (1) is $y_C = Ae^x + Be^{3x}$, where A, B are two arbitrary constants.

To find P.I., since the right hand side of (1) is a polynomial of degree 2, therefore, we assume, the P.I. as $y_P = c_1x^2 + c_2x + c_3$, where c_1, c_2 and c_3 are constants to be determined from the equation $(D^2 - 4D + 3)y_P = 12x^2 + x - 6$ ... (3)

From (3) we have, $(D^2 - 4D + 3)(c_1x^2 + c_2x + c_3) = 12x^2 + x - 6$

i.e., $D^2(c_1x^2 + c_2x + c_3) - 4D(c_1x^2 + c_2x + c_3) + 3(c_1x^2 + c_2x + c_3) = 12x^2 + x - 6$

i.e., $2c_1 - 4(2c_1x + c_2) + 3c_1x^2 + 3c_2x + 3c_3 = 12x^2 + x - 6$

i.e., $3c_1x^2 + (3c_2 - 8c_1)x + 2c_1 - 4c_2 + 3c_3 = 12x^2 + x - 6$

Equating the like powers of x, we have,

$$3c_1 = 12,\ (3c_2 - 8c_1) = 1,\ 2c_1 - 4c_2 + 3c_3 = -6$$

i.e., $c_1 = 4$, $c_2 = \frac{1}{3}(1 + 32) = 11$, $c_3 = \frac{1}{3}(-6 + 44 - 8) = 10$

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Therefore the P.I. is $y_P = 4x^2 + 11x + 10$

Thus the general solution of the given differential equation is

$y = y_C + y_P = Ae^x + Be^{3x} + 4x^2 + 11x + 10$, where A, B are arbitrary constants can be determined with the help of given initial conditions.

Now, $\frac{dy}{dx} = A + 3Be^{3x} + 8x + 11$

From the condition, $y = 6$, when $x = 0$, we have, $A + B + 10 = 6$ *i.e.*, $A + B = -4$ ... (4)

Again from the condition $\frac{dy}{dx} = 11$ when $x = 0$, we have,

$11 = A + 3B + 11$ *i.e.*, $A + 3B = 0$ ... (5)

Solving (4) and (5), we have $A = -6$ and $B = 2$.

Therefore we have the required general solution for the particular case is as

$y = -6e^x + 2e^{3x} + 4x^2 + 11x + 10$

11. **Solve $(D^2 - 4D + 4)y = (x + x^3)e^{2x}$ by method of undetermined coefficient.** **[CH 2002]**

Solution Here the given differential equation is

$(D^2 - 4D + 4)y = (x + x^3)e^{2x}$, $D \equiv \frac{d}{dx}$ ... (1)

The reduced equation of (1) is $(D^2 - 4D + 4)y = 0$ ... (2)

Let $y = ce^{mx}$ $(c \neq 0)$ be a trial solution of (2)

Then A.E. becomes $m^2 - 4m + 4 = 0$ *i.e.*, $(m - 2)^2 = 0$ solving $m = 2, 2$

Since $m = 2$ is a double root of the A.E., therefore the C.F. is

$y_C = (A + Bx)e^{2x}$, A, B are two arbitrary constants.

To find P.I., since 2 is a double root of A.E., so we choose the P.I.,

$y_P = x^2e^{2x}(c_1x^3 + c_2x^2 + c_3x + c_4)$, where c_1, c_2, c_3, c_4 are constants can be found from

Solving the equation $(D^2 - 4D + 4)y_P = (x + x^3)e^{2x}$ ... (3)

Simplifying we have, $D^2y_P = D^2(x^2e^{2x}(c_1x^3 + c_2x^2 + c_3x + c_4))$

$= 2e^{2x}(1 + 4x + 2x^2)(c_1x^3 + c_2x^2 + c_3x + c_4) + 4xe^{2x}(1 + x)(3c_1x^2 + 2c_2x + c_3) + 2x^2e^{2x}(3c_1x + c_2)$

and $Dy_P = D(x^2e^{2x}(c_1x^3 + c_2x^2 + c_3x + c_4)$

$= 2(x + x^2)e^{2x}(c_1x^3 + c_2x^2 + c_3x + c_4) + x^2e^{2x}(3c_1x^2 + 2c_2x + c_3)$

Substituting D^2y_P, Dy_P and y_P in (3) and equating the corresponding coefficients and solving we have, $c_1 = \frac{1}{20}$, $c_2 = 0$, $c_3 = \frac{1}{6}$, $c_4 = 0$

Thus the P.I. is $y_P = \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x}$

Hence the required general solution is $y = y_C + y_P$

i.e., $y = \left(A + Bx + \frac{1}{6}x^3 + \frac{1}{20}x^5\right)e^{2x}$, where A and B are two arbitrary constants.

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Exercise

- **Solve the following differential equations (1-20) by method of undetermined coefficients :**

1. $(D^2 - 3D + 2)y = 2e^{3x}$
2. $(D^2 + D + 1)y = e^{2x}$
3. $y'' - 4y' + 4y = e^{2x}$
4. $(D^2 - 2D + 1) = e^{x}$
5. $(D^2 - 4D + 3)y = 2e^{3x}$
6. $(D^2 + 1)y = \cos 2x$
7. $(D^2 - 2D + 5)y = 10\sin x$
8. $(D^2 - 2D + 2)y = 5\cos x$
9. $(D^2 + 4)y = \sin 2x$
10. $y'' + 2y' + y = x^2 + 2x$
11. $(D^2 - 4)y = x^2$
12. $(D^2 + 2D)y = x^2$
13. $(D^2 - 2D + 1)y = e^x + x^2$
14. $(D^2 - 6D + 7)y = e^x + e^{-x}$
15. $y'' + y' - 2y = x + \sin x$
16. $(D^2 - 2D + 1)y = e^{3x}\cos x$
17. $(D^2 - 2D + 1)y = x^2e^x$.
18. $(D^2 - 7D + 6)y = (x - 2)e^x$ [CH 2000]
19. $(D^2 - 3D)y = x + e^x\sin x$ [CH 2004]
20. $(D^2 + D - 6)y = 10e^{2x} - 18e^{3x} - 6x - 11$ [CH 1997, 2001]
21. Solve by method of underdetermined coefficient $(D^2 - 7D + 6)y = 2e^{3x}$, given that $y = 1$, $Dy = 0$ at $x = 0$ [BH 1986]
22. Solve by method of undetermined coefficient, the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$; given that $y = \frac{3}{8}$, $\frac{dy}{dx} = 1$ at $x = 0$ [CP 1995]

Answers

1. $y = Ae^x + Be^{2x} + e^{3x}$
2. $y = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{7}e^{2x}$
3. $y = (A + Bx)e^{2x} + \frac{1}{2}x^2e^{2x}$
4. $y = (A + Bx)e^x + \frac{1}{2}x^2e^x$
5. $y = Ae^x + Be^{3x} + xe^{3x}$
6. $y = A\cos x + B\sin x - \frac{1}{3}$
7. $y = e^x(A\cos 2x + B\sin 2x) + 2\sin x + \cos x$
8. $y = (A\cos x + B\sin x)e^x + \cos x - 2\sin x$
9. $y = A\cos 2x + B\sin 2x - \frac{x}{4}\cos x$
10. $y = (A + Bx)e^{-x} + x^2 - 2x + 2$
11. $y = Ae^{2x} + Be^{-2x} - \frac{1}{8}(2x^2 + 1)$
12. $y = (A + Bx)e^{-x} + x^2 - 2x + 2$
13. $y = (A + Bx + \frac{1}{2}x^2)e^x + x^2 + 4x + 6$
14. $y = Ae^{(3+\sqrt{2})x} + Be^{(3-\sqrt{2})x} + \frac{1}{2}e^x + \frac{1}{14}e^{-x}$
15. $y = Ae^x + Be^{-2x} - \frac{1}{4}(2x + 1) - \frac{1}{10}(3\sin x + \cos x)$
16. $y = (A + Bx)e^x + \frac{1}{25}(3\cos x + 4\sin x)e^{3x}$
17. $y = \left(A + Bx + \frac{x^4}{12}\right)e^x$
18. $y = Ae^x + Be^{6x} - \frac{1}{5}\left(\frac{x^2}{2} - \frac{9}{5}x\right)e^x$
19. $y = A + Be^{3x} - \frac{1}{6}x^2 - \frac{1}{9}x + \frac{e^x}{10}(\cos x - 3\sin x)$
20. $y = Ae^{2x} + Be^{-3x} + 2xe^{2x} - 3e^{3x} + x + 2$
21. $y = \frac{7}{5}e^x - \frac{1}{3}e^{3x} - \frac{1}{15}e^{6x}$
22. $y = \frac{1}{2}xe^{2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$

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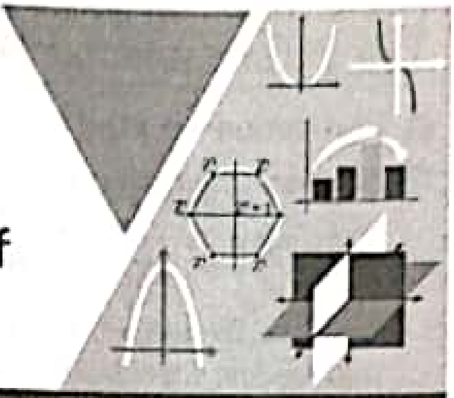
10 CHAPTER

Wronskian and Method of Variation of Parameters



Introduction

In this chapter we will discuss a method to determine the general solution of second order linear differential equation. We consider the standard form of a second order linear ordinary differential equation with constant co-efficients as follows :

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = \phi(x) \qquad \dots (1)$$

where p_1 and p_2 are constants and $\phi(x)$ is a functions of x only.

We have already determined the solution of corresponding reduced form of the equation (1)

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = 0 \qquad \dots (2)$$

which is known as the complementory function or C.F., containing two arbitrary constants.

However the linear independence of n-functions (provided those function are continuous and differentiable up to $(n-1)$th order) can be easily checked by finding the **Wronskian** of those function.

10.1 Wronskian

Suppose that $y_1(x), y_2(x), \dots, y_n(x)$ are n functions of a single variable x and each of those n functions assumes any order differentiation up to $(n-1)$th order. Then the Wronskian of $y_1(x), y_2(x), \dots, y_n(x)$ is denoted by $W(y_1, y_2, \dots, y_n)$ and is defined by :

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1'' & y_2'' & \dots & y_n'' \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \qquad \dots (3)$$

In the following theorem the inter-relation between linear independence and Wronskian is stated.

Theorem : ***The functions $y_1(x), y_2(x), \dots, y_n(x)$ are linearly independent on some closed interval $[a, b]$ if and only if their Wronskian $W(y_1, y_2, \dots, y_n)$ is purely non-zero for each $x \in [a, b]$.***

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Illustration-1 : Find the Wronskian of $y_1(x) = x^2$ and $y_2(x) = x^3$.

Solution : From the definition of Wronskian we have, for $y_1 = x^2$ and $y_2 = x^3$,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \ (\neq 0)$$

Thus the Wronskian of x^2 and x^3 is non-zero and so the functions $y_1 = x^2$ and $y_2 = x^3$ are independent functions.

In this regard we may state the following two properties :

- **Property-1 :** The function $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent if and only if their Wronskian $W(y_1, y_2, \ldots, y_n) \neq 0$. On the other hand if $W(y_1, y_2, \ldots, y_n) = 0$; then the functions $y_1, y_2, \ldots, y_n$ are dependent.

10.2 Relation between Wronskian and Differential Equation

Consider a second order differential equation with constant coefficient presented in equation (2). Let y_1 and y_2 be two solutions of the second order differential equation (2). Then we have the following property.

- **Property-2 :** In any interval $[a, b]$, the Wronskian $W(y_1, y_2)$ of the functions y_1 and y_2 is either identically zero or never becomes zero. However if $W(y_1, y_2) = 0$ then y_1 and y_2 are linearly dependent anf if $W(y_1, y_2) \neq 0$ then y_1 and y_2 are linearly independent. In last case the general solution of the differential equation (2) is $y(x) = Ay_1(x) + By_2(x)$, where A and B are two arbitrary constants. Certainly this property can be extended to a third and higher order differential equation also.

Notes

1. If y_1 and y_2 are two independent solutions of the equation (2) then the Wronskian $W(y_1, y_2)$ can be determined from $W(y_1, y_2) = ce^{-\int p_1 dx}$, where c is a constant and p_1 is the coefficient of $\frac{dy}{dx}$ in (2).
2. If $y = y_1(x)$ is is a solution of differential equation (2), then other independent solution $y = y_2(x)$ is given by $y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2} dx$, where $W(y_1, y_2)$ is the Wronskian satisfies the the differential equation $\frac{dW(y_1, y_2)}{dx} + p_1 W(y_1, y_2) = 0$.

10.3 Application of Wronskian in Second and Higher Order Linear Differential Equation

Let y_1 and y_2 be two solutions of the differential equation (2). Now if the Wronskian of those two solutions *i.e.*, $W(y_1, y_2)$ where

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is non-zero then we may claim those two solutions are ind[illegible]

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this case the general solution of the equation (2) will be $y = c_1y_1 + c_2y_2$, where c_1 and c_2 two arbitrary constants.

Illustration-2 : Consider the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

We see by inspection or using A.E., $y_1 = e^x$ and $y_2 = e^{2x}$ be two solutions of that equation.

Now $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^xe^{2x} - e^xe^{2x} = e^{3x} \neq 0$

Therefore the solutions $y_1 = e^x$ and $y_2 = e^{2x}$ are independent and the general solution is thus $y = c_1e^x + c_2e^{2x}$ where c_1 and c_2 are two arbitrary constants.

Illustration-3 : Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$.

We see that the roots of the auxiliary equations are 1 and 1 but if we choose $y_1 = e^x$ and $y_2 = e^x$ then we have a linearly dependent solution since

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^x \\ e^x & e^x \end{vmatrix} = 0$

Still we observe that, if we consider $y_2 = xe^x$ (instead of considering $y_2 = e^x$), then also y_2 satisfies the given differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. In this case the Wronskian $W(y_1, y_2)$ is given by $W(y_1, y_2) \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x} \neq 0$

Therefore $y_1 = e^x$ and $y_2 = xe^x$ are two independent solutions and therefore the general solution is given by $y = c_1y_1 + c_2y_2$

i.e., $y = c_1e^x + c_2xe^x = (c_1 + c_2x)e^x$, where c_1 and c_2 are two arbitrary constants.

10.4 Variation of Parameter Technique to Solve a Second Order Linear Differential Equation

We recall the second order linear differential equation with constant coefficients given in (1).

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2y = \phi(x) \quad \text{... (1)}$$

where p_1 and p_2 are constants and $\phi(x)$ is a function of x only.

Below we describe the steps to solve the differential equation (1) by Variation of parameter technique :

- **Step-1 :** First find two linearly independent solutions y_1 and y_2 for the reduced form of the differential equation (1).
  Then the general solution of the corresponding reduced equation (C.F.) is given by $y_C = c_1y_1 + c_2y_2$, where c_1 and c_2 are two arbitrary constants. Also the linear independency of the solutions y_1 and y_2 can be checked by finding Wronskian of them.

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- **Step-2 :** Let the complete solution or the general solution of the differential equation (1) be $y(x) = u(x)y_1(x) + v(x)y_2(x)$ ... (2)
  which is formed by replacing two arbitrary constants c_1 and c_2 by two functions $u(x)$ and $v(x)$ which are to be determined in the following steps.
- **Step-3 :** Differentiating (2) w.r. to x we have,
  $$\frac{dy}{dx} = \left(u(x)\frac{dy_1}{dx} + v(x)\frac{dy_2}{dx}\right) + \left(y_1(x)\frac{du}{dx} + y_2(x)\frac{dv}{dx}\right) \quad \text{... (3)}$$
  The functions $u(x)$ and $v(x)$ are to be chosen in such a way that
  $$y_1(x)\frac{du}{dx} + y_2(x)\frac{dv}{dx} = 0 \quad \text{... (4)}$$
  Then (3) reduces to $\frac{dy}{dx} = u(x)\frac{dy_1}{dx} + v(x)\frac{dy_2}{dx}$ ... (5)
- **Step-4 :** Again differentiating (5) w.r. to x we have,
  $$\frac{d^2y}{dx^2} = u(x)\frac{d^2y_1}{dx^2} + v(x)\frac{d^2y_2}{dx^2} + \left(\frac{dy_1}{dx}\cdot\frac{du}{dx} + \frac{dy_2}{dx}\cdot\frac{dv}{dx}\right) \quad \text{... (6)}$$
  Now substituting the values of $\frac{dy}{dx}$ (from (5)) and $\frac{d^2y}{dx^2}$ (from (6)) in the differential equation (1) and simplifying we have, $\frac{dy_1}{dx}\cdot\frac{du}{dx} + \frac{dy_2}{dx}\cdot\frac{dv}{dx} = \phi(x)$ ... (7)
- **Step-5 :** Since $y_1(x)$, $y_2(x)$ and $\phi(x)$ are known functions of x only therefore from (5) and (7) we can determine the unique values of $\frac{du}{dx}$ and $\frac{dv}{dx}$. Clearly both of them must be functions of x only. Let us assume,
  $\frac{du}{dx} = F_1(x)$ and $\frac{dv}{dx} = F_2(x)$.
- **Step-6 :** On integration we have, $u(x) = \int F_1(x)dx + A$ and $v(x) = \int F_2(x)dx + B$, where A and B are two arbitrary constants.
- **Step-7 :** Substituting the values of $u(x)$ and $v(x)$ in (2) we have the required general solutions which is as follows :
  $$y = Ay_1(x) + By_2(x) + y_1(x)\int F_1(x)dx + y_2(x)\int F_2(x)dx \quad \text{... (8)}$$
  where A and B are two arbitrary constants and $F_1(x)$, $F_2(x)$ are the solutions of the differential equations $\frac{du}{dx} = F_1(x)$ and $\frac{dv}{dx} = F_2(x)$ respectively.

The above discussed steps (step-1–step-7) are required to solve a second order linear differential equation by variation of parameters technique and the whole method to determine the general solution is known as variation of parameters.

Notes

1. Instead of second order linear differential equation with constant coefficients, if the form is homogeneous, then by suitable transformations, first we need to transform it second order linear differential equation with constant coefficients and then follow above mentioned steps to determine the general solution.
2. Similarly the method can be extended to third and higher order case also.

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Alternative Method

If y_1 and y_2 be two independent solutions of the equation (2) then the general solution of the equation (1) will be $y = y_C + y_P$, where y_C = C.F. of (1) = $Ay_1 + By_2$ and y_P = P.I. of (1) $= uy_1 + vy_2$, where A, B are two arbitrary constants and u, v can be determined from the following formula : $u = -\int \frac{y_2\phi(x)}{W}\,dx$ and $v = \int \frac{y_1\phi(x)}{W}\,dx$; W being the Wronskian of y_1 and y_2.

In next example, we illustrate the above method :

Illustration-4 : Consider the differential equation $\frac{d^2y}{dx^2} + a^2y = \sin ax$ ... (9)

Let $y = e^{mx}$ be a trial solution of the reduced form of (9).

Then the auxiliary equation is $m^2 + a^2 = 0$ giving $m = \pm ai$

Therefore, C.F. = $A\cos ax + B\sin ax$, where A and B are two arbitrary constants.

Let $y_1 = \cos ax$ and $y_2 = \sin ax$.

Now, Wronskian of y_1 and y_2 is $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0$

Thus the functions y_1 and y_2 are linearly independent.

Let the P.I. be $y_P = uy_1 + vy_2 = u\cos ax + v\sin ax$ ... (2)

where $u = -\int \frac{y_2\phi(x)}{W}\,dx$ and $v = \int \frac{y_1\phi(x)}{W}\,dx$, where $\phi(x) = \sin ax$

$$\text{Therefore, } u = -\int \frac{\sin ax \sin ax}{a}\,dx = -\frac{1}{2a}\int (1-\cos 2ax)\,dx = -\frac{1}{2a}\left(x - \frac{\sin 2ax}{2a}\right)$$

$$v = \int \frac{\cos ax \sin ax}{a}\,dx = \frac{1}{2a}\int \sin 2ax\,dx = -\frac{1}{4a^2}\cos 2ax$$

$$\text{Therefore, P.I.} = -\frac{1}{2a}\left(x\cos ax - \frac{\sin 2ax \cos ax}{2a}\right) - \frac{1}{4a^2}\cos 2ax \sin ax$$

$$= -\frac{1}{2a}x\cos ax + \frac{1}{4a^2}[\sin 2ax \cos ax - \cos 2ax \sin ax]$$

$$= -\frac{1}{2a}x\cos ax + \frac{1}{4a^2}\sin ax$$

Therefore the general solution is y = C.F. + P.I.

$$= A\cos ax + B\sin ax - \frac{1}{2a}x\cos ax + \frac{1}{4a^2}\sin ax$$

$= A\cos ax + B_1\sin ax - \frac{1}{2a}x\cos ax$, where $B_1 = \left(B + \frac{1}{4a^2}\right)$ and A are two arbitrary constants.

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Illustrative Examples

1. **Show that $y_1(x) = \sin ax$ and $y_2(x) = \cos ax$ are the two independent soluti[ons] of the differential equation $\frac{d^2y}{dx^2} - a^2y = 0$.**

Solution Here the given two functions are $y_1(x) = \sin ax$ and $y_2(x) = \cos ax$

Now the Wronskian of y_1 and y_2 is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin ax & \cos ax \\ a\cos ax & -a\sin ax \end{vmatrix} = -a(\sin^2 ax + \cos^2 ax) = -a \neq 0$$

Since the Wronskian of $y_1(x)$ and $y_2(x)$ is non-zero,
therefore $y_1(x) = \sin ax$ and $y_2(x) = \cos ax$ are linearly independent.

Further, we have, $\frac{d^2y_1}{dx^2} - a_2y_1 = 0$ and $\frac{d^2y_2}{dx^2} - a^2y_2 = 0$ and so $y_1(x) = \sin ax$ and $y_2(x) = \cos ax$ are two independent solutions of the differential equation $\frac{d^2y}{dx^2} - a^2y = 0$.

Therefore the general solution of the differential equation is $y(x) = A\sin ax + B\cos ax$, whose A and B are two arbitrary constants.

2. **Show that the functions $y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = x^2$ are independent. Also find the differential equation whose three independent solutions are 1, x and x^2.**

Solution The Wronskian of 1, x, and x^2 is

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

Since the Wronskian of the function $y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = x^2$ is non-zero, to three function are independent. Further the general solution of the differential equation whose three independent are $y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = x^2$ can be written as :

$y(x) = A + Bx + Cx^2$, whose A, B, C are three arbitrary constants. ... (1)

Now differentiating (1), thrice with respect to x, we have,

$$\frac{dy}{dx} = B + 2Cx$$

$$\frac{d^2y}{dx^2} = 2C$$

$\frac{d^3y}{dx^3} = 0$, this is the required differential equation wh[...] inde[...] solutions are $y_1(x) = 1$, $y_2(x) = x$, and $y_3(x) = x^2$.

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3. **Given that $f(x)$, $g(x)$ and $h(x)$ are solutions of $y'' + p(x)y' + q(x)y = 0$. Show that $\begin{vmatrix} f'' & f' & f \\ g'' & g' & g \\ h'' & h' & h \end{vmatrix} = 0$. What can you say about these three solutions.** [BH 2006]

Solution Here the given differential equation is

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{... (1)}$$

Since $f(x)$, $g(x)$ and $h(x)$ are three solutions of (1) then we have from (1)

$$f''(x) + p(x)f'(x) + q(x)f(x) = 0 \qquad \text{... (2)}$$

$$g''(x) + p(x)g'(x) + q(x)g(x) = 0 \qquad \text{... (3)}$$

$$h''(x) + p(x)h'(x) + q(x)h(x) = 0 \qquad \text{... (4)}$$

Eliminating $p(x)$ and $q(x)$ from (2), (3) and (4) we have

$$\begin{vmatrix} f'' & f' & f \\ g'' & g' & g \\ h'' & h' & h \end{vmatrix} = 0 \quad \textbf{[Proved]}$$

From the above result it can be concluded that the solutions $f(x)$, $g(x)$ and $h(x)$ are linearly dependent. Hence for a second order differential equation there can not have three linearly independent solution.

4. **Show that $y_1(x) = e^x$ and $y_2(x) = xe^x$ are two independent functions. Hence find a differential equation whose two independent solutions are e^x and xe^x.**

Solution Here the given two functions are : $y_1(x) = e^x$ and $y_2(x) = xe^x$.

The Wronskian is given by $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}$

$$= e^{2x}(x + 1) - xe^{2x} = e^{2x} \neq 0$$

Since the Wronskian of $y_1(x) = e^x$ and $y_2(x) = xe^x$ is non-zero, then the given functions are independent. Therefore the general solution of the differential equation whose two solutions are $y_1(x) = e^x$ and $y_2(x) = xe^x$ is

$y(x) = Ae^x + Bxe^x$, where A and B are two arbitrary constants. ... (1)

Now differentiating (1) twice with respect to x, we have,

$$\frac{dy}{dx} = Ae^x + Bxe^x + Be^x = y + Be^x \qquad \text{... (2)}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + Be^x = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) \quad \text{[From (2)]}$$

i.e., $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$, which is the required differential equations.

5. **Find the Wronskian of x, $2x + 3$ and $4x + 9$. What do we conclude from the Wronskian?**

Solution Let $y_1(x) = x$, $y_2(x) = 2x + 3$, $y_3(x) = 4x + 9$.

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Now the Wronskian is given by :

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & 2x+3 & 4x+9 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Hence the Wronskian of given three variables is zero. From this value, it may be concluded that the given three functions are linearly dependent. It can be established since

$$4x + 9 = 3(2x + 3) - 2x \text{ i.e., } 2x - 3(2x + 3) + (4x + 9) = 0$$

6. **Solve, by method of variation of parameters, the differential equation** $\dfrac{d^2y}{dx^2} + 4y = 4\tan 2x$ **[BU 1999, 2004, 2010; VU 1991]**

Solution Here the given differential equation is

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x \qquad \text{... (1)}$$

The reduced equation of (1) becomes $\dfrac{d^2y}{dx^2} + 4y = 0$ ... (2)

Let $y = ce^{mx}$ be a trial solution of (2). Then A.E. is $m^2 + 4 = 0$ giving $m = \pm 2i$

Therefore the complementary function of (1) is

$y_C = A\cos 2x + B\sin 2x$, A, B are two arbitrary constants.

Let $y_1 = \cos 2x$ and $y_2 = \sin 2x$

Now Wronskian of y_1 and y_2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2 \neq 0$$

Therefore the functions, y_1 and y_2 are linearly independent. So, we assume the complete general solution of the equation (1) is

$$y = uy_1 + vy_2 = u\cos 2x + v\sin 2x \qquad \text{... (3)}$$

where u and v, two functions of x are to be determined.

Now, $\dfrac{dy}{dx} = \cos 2x\dfrac{du}{dx} + \sin 2x\dfrac{dv}{dx} - 2u\sin 2x + 2v\cos 2x$ ... (4)

We choose u and v in such a way that

$$\cos 2x\frac{du}{dx} + \sin 2x\frac{dv}{dx} = 0 \qquad \text{... (5)}$$

So that we have, $\dfrac{dy}{dx} = -2u\sin 2x + 2v\cos 2x$ ...

Also $\dfrac{d^2u}{dx^2} = -4u\cos 2x - 4v\sin 2x - 2\sin 2x\dfrac{du}{dx} + 2\cos 2x\dfrac{dv}{dx}$

We choose u and v in such a way that,

$$-2\sin 2x\frac{du}{dx} + 2\cos 2x\frac{dv}{dx} = 4\tan 2x$$

$$\text{i.e., } -2\sin 2x\frac{du}{dx} + \cos 2x\frac{dv}{dx} = 2\tan 2x$$

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Solving (5) and (7) we get,

$$\frac{du}{dx} = -\frac{2\sin^2 2x}{\cos 2x} \text{ and } \frac{dv}{dx} = 2\sin 2x$$

Integrating we get,

$u(x) = A - 2\int \frac{1 - \cos^2 2x}{\cos 2x}\,dx$, where A is an arbitrary constant

$$= A - 2\int \sec 2x\,dx + 2\int \cos 2x\,dx = A - \log(\sec 2x + \tan 2x) + \sin 2x$$

Also $v(x) = \int 2\sin 2x\,dx + B = B - \cos 2x$, where B is an arbitrary constant.

Therefore the required general solution of the given differential equation is

$$y(x) = u(x)y_1 + v(x)y_2 = A\cos 2x - \cos 2x \log(\sec 2x + \tan 2x) + B\sin 2x$$

where A and B are two arbitrary constants.

7. Solve : $\frac{d^2y}{dx^2} + a^2y = \sec ax$ by method of variation of parameters.

Solution Here the given differential equation is $\frac{d^2y}{dx^2} + a^2y = \sec ax$ ... (1)

The reduced equation of (1) becomes $\frac{d^2y}{dx^2} + a^2y = 0$ ... (2)

Let $y = ce^{mx}$ be a trial solution of (2). Then the A.E. is $m^2 + a^2 = 0$ giving $m = \pm ia$

Therefore the C.F. of (1) is $y_C = c_1\cos ax + c_2\sin ax$ ... (1)

where c_1 and c_2 are two arbitrary constants.

Let $y_1 = \cos ax$ and $y_2 = \sin ax$

Now Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a \neq 0$$

Therefore the solutions $y_1 = \cos ax$ and $y_2 = \sin ax$ are independent solutions. So let us take the general solution of the given differential equation as

$$y = u(x)y_1 + v(x)y_2 = u(x)\cos ax + v(x)\sin ax \quad \text{... (2)}$$

where $u(x)$ and $v(x)$ are two functions and required to be determined.

Now, $\frac{dy}{dx} = -au(x)\sin ax + av(x)\cos ax + \cos ax\frac{du}{dx} + \sin ax\frac{dv}{dx}$ ... (3)

We choose $u(x)$ and $v(x)$ in such a way that,

$$\cos ax\frac{du}{dx} + \sin ax\frac{dv}{dx} = 0 \quad \text{... (4)}$$

Thus from (3) we have, $\frac{dy}{dx} = -au\sin ax + av\cos ax$

Therefore, $\frac{d^2y}{dx^2} = -a^2u\cos ax - a^2v\sin ax - a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx}$ ... (5)

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Substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we have,

$$a^2u\cos ax + a^2v\sin ax - a^2u\cos ax - a^2v\sin ax - a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} = \sec ax$$

i.e., $-a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} = \sec ax$ ... (6)

Solving (4) and (6) we have, $\frac{du}{dx} = -\frac{1}{a}\tan ax$ and $\frac{dv}{dx} = \frac{1}{a}$

Integrating we get,

$u(x) = A + \frac{1}{a^2}\log\cos ax$ and $v(x) = B + \frac{x}{a}$, where A and B are two arbitrary constants.

Substituting the values of $u(x)$ and $v(x)$ in (2), we have the required general solution

$$y(x) = A\cos ax + B\sin ax + \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax$$

where A and B are two arbitrary constants.

8. **Solve by method of variation of parameters $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$** [CH '95]

Solution Here the given differential equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$ ...(1)

The reduced equation of (1) is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ ... (2)

Let $y = ce^{mx}$ be a trial solution of (2). Then the A.E. of (2) is $m^2 - 3m + 2 = 0$

i.e., $(m-1)(m-2) = 0$ giving $m = 1, 2$

Therefore, the C.F. of (1) becomes,

$y_C = c_1e^x + c_2e^{2x}$, where c_1 and c_2 are two arbitrary constants.

Now let $y_1 = e^x$, $y_2 = e^{2x}$, clearly $y_1 = e^x$, $y_2 = e^{2x}$ are two independent solutions since Wronskian of y_1 and y_2,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^xe^{2x} - e^xe^{2x} = e^xe^{2x} \neq 0$$

To solve the equation by method of variation of parameters, we choose the general solution as $y(x) = u(x)y_1 + v(x)y_2 = u(x)e^x + v(x)e^{2x}$

where $u(x)$ and $v(x)$ are two functins to be determined. Now we have ... (2)

$$\frac{dy}{dx} = u(x)e^x + 2v(x)e^{2x} + e^x\frac{du}{dx} + e^{2x}\frac{dv}{dx} \quad ... (3)$$

We choose $u(x)$ and $v(x)$ in such a way that,

$$e^x\frac{du}{dx} + e^{2x}\frac{dv}{dx} = 0 \quad ... (4)$$

Then from (3), we have $\frac{dy}{dx} = u(x)e^x + 2v(x)e^{2x}$ ... (5)

Differentiating (5), we have $\frac{d^2y}{dx^2} = u(x)e^x + 4v(x)e^{2x} + e^x\frac{du}{dx} + 2e^{2x}\frac{dv}{dx}$ ... (6)

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Using (5) and (6) in (1) we have,

$$u(x)e^x + 4v(x)e^{2x} + e^x\frac{du}{dx} + 2e^{2x}\frac{dv}{dx} - 3(u(x)e^x + 2v(x)e^{2x}) + 2(u(x)e^x + v(x)e^{2x}) = \frac{e^x}{1+e^x}$$

i.e., $e^x\frac{du}{dx} + 2e^{2x}\frac{dv}{dx} = \frac{e^x}{1+e^x}$ ... (7)

Subtracting (4) from (7) we have, $e^{2x}\frac{dv}{dx} = \frac{e^x}{1+e^x}$

i.e., $\frac{dv}{dx} = \frac{1}{e^x(1+e^x)} = \frac{(1+e^x)-e^x}{e^x(1+e^x)} = \frac{1}{e^x} - \frac{1}{1+e^x} = e^{-x} - \frac{e^{-x}}{1+e^{-x}}$

Integrating we have $v(x) = -e^{-x} + \log(1 + e^{-x}) + B$, where B is an arbitrary constant.

Also we have, $e^x\frac{du}{dx} = -\frac{e^x}{1+e^x}$ i.e., $\frac{du}{dx} = -\frac{e^{-x}}{1+e^{-x}}$

Integrating we get, $u(x) = \log(1 + e^{-x}) + A$, where A is an arbitrary constant.

Putting u and v in (2), the general solution is

$$y(x) = Ae^x + Be^{2x} - e^x + (e^x + e^{2x})\log(1 + e^{-x}) = A_1e^x + Be^{2x} + (e^x + e^{2x})\log(1 + e^{-x}),$$

where $A_1 = A - 1$ and B are two arbitrary constants.

9. Solve $(D^2 - 2D + 1)y = e^{2x}(e^x + 1)^{-2}$ by variation of parameters where $D \equiv \frac{d}{dx}$

Solution Here the given differential equation is

$$(D^2 - 2D + 1)y = e^{2x}(e^x + 1)^{-2} \quad \text{... (1)}$$

The reduced equation of (1) is $(D^2 - 2D + 1)y = 0$ ... (2)

Let $y = ce^{mx}$ be a trial solution of (2). Then the A.E. is $m^2 - 2m + 1 = 0$

i.e., $(m - 1)^2 = 0$ giving $m = 1, 1$

Therefore, the C.F. of the given differential equation is

$$y = (c_1 + c_2x)e^x \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

To check the independency of the solutions $y_1(x) = e^x$ and $y_2(x) = xe^x$, we consider the Wronskian of y_1 and y_2 as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x} \neq 0$$

Since $y_1(x)$ and $y_2(x)$ are two independent solutions, therefore, we choose the general solution of (1) as

$$y(x) = u(x)y_1(x) + v(x)y_2(x) = e^xu(x) + xe^xv(x) \quad \text{... (3)}$$

where $u(x)$ and $v(x)$ are two functions to be determined

From (3) we have,

$$\frac{dy}{dx} = e^xu(x) + (e^x + xe^x)v(x) + e^x\frac{du}{dx} + xe^x\frac{dv}{dx} \quad \text{... (4)}$$

Let us choose $u(x)$ and $v(x)$ in such a way that,

$$e^x\frac{du}{dx} + xe^x\frac{dv}{dx} = 0 \quad \text{... (5)}$$

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11 CHAPTER

Simultaneous Differential Equation

Introduction

Till now we have solved the ordinary differential equation containing one independent and one dependent variable. In an ordinary differential equation there will always be one independent variable and one dependent variable. A system of simultaneous differential equation is a system which contains more than one differential equations with one independent variable and more than one dependent variables. To keep our discussions within the limit of the syllabus, we consider a system of differential equations containing two equations. However a system containing more than two equations also would be a system of simultaneous differential equations. More specifically we are interested to find the solution of simultaneous linear differential equations only and solution procedure for non-linear simultaneous differential equations are omitted here.

There are mainly two types of simultaneous linear differential equations which are given below :

- **Type-1 :**

$$\left.\begin{aligned} \frac{dx}{dt} + f_1(x, y, t) = 0 \\ \frac{dx}{dt} + f_2(x, y, t) = 0 \end{aligned}\right\} \quad ...(1)$$

Where f_1 and f_2 are two functions of dependent variables x, y and independent variable t.

- **Type-2 :**

$$\left.\begin{aligned} P_1 dx + Q_1 dy + R_1 dz = 0 \\ P_2 dx + Q_2 dy + R_2 dz = 0 \end{aligned}\right\} \quad ...(2)$$

Where P_1, Q_1, R_1, P_2, Q_2 and R_2 are functions of x, y and z or constants and among x, y, z one is independent variable and other two are dependent variables.

11.1 Solution Procedure of Type-1 Differential Equation

The general form the Type-1 simultaneous differential equation with linear from can be presented as :

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$$\left.\begin{aligned}\frac{dx}{dt}+a_1x+b_1y=\phi_1(t)\\ \frac{dx}{dt}+a_2x+b_2y=\phi_2(t)\end{aligned}\right\} \quad ...(3)$$

Where a_1, a_2, b_1, b_2 are constants only and $\phi_1(t)$, $\phi_2(t)$ are functions of t only. Since we are aiming to solve simultaneous differential equations of linear type therefore, here all of a_1, a_2, b_1 and b_2 should be constants only. Again if both of $\phi_1(t)$ and $\phi_2(t)$ are constants then the system (3) can be re-written as :

$$\left.\begin{aligned}\frac{dx}{dt}+a_1x+b_1y=0\\ \frac{dy}{dt}+a_2x+b_2y=0\end{aligned}\right\} \quad ...(4)$$

The from of the system of differential equation (4) is known as the **homogeneous form of simultaneous linear differential equation** and the system (3) is known as **non-homogeneous simultaneous differential equation.** Introducing the differential operator $D \equiv \frac{d}{dt}$, the system (3) and (4) can be re-written as :

$$\left.\begin{aligned}(D+a_1)x+b_1y=\phi_1(t)\\ a_2x+(D+b_2)y=\phi_2(t)\end{aligned}\right\} \quad ...(5)$$

and
$$\left.\begin{aligned}(D+a_1)x+b_1y=0\\ a_2x+(D+b_2)y=0\end{aligned}\right\} \quad ...(6)$$

Below we discuss the methods to find the general solution of equations in homogenous from (6) [or the system (4)] and then we discuss the methods for solving corresponding non-homogeneous system of differential equation (5) [or the system (3)]. Further it should be noted that, since the system of differential equations (3), (4), (5), (6) contain two dependent variables and one independent variables, so the general solution should contain exactly two arbitrary constants.

- **Method-1** : To solve the homogeneous system (6) [or system (4)].
  - **Step-1** : Eliminate either x or y from the two equations of the system (6). If we eliminate y between those two, we have,

$$\left.\begin{aligned}(D+a_1)(D+b_2)x-a_2b_1x=0\\ i.e.,\quad D^2x+(a_1+b_2)Dx+(a_1b_2-a_2b_2)x=0\end{aligned}\right\} \quad ...(7)$$

Equation (7) is a second order differential equation with constant coefficients and whose general solution, $x = x(t)$ can easily be found. Also that general solution contains two arbitrary constants since equation (7) is a second order differential equation. Then substituting the value $x = x(t)$ in first equation of (6) we obtain.

$$y(t)=-\frac{1}{b_1}[a_1x(t)+x'(t)]$$

Which will be general solution of $y(t)$. Combining these two values, we have required general solution.

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Notes

1. Instead of eliminating y between two equations of (6), if we eliminate x, then we obtain a second order differential equation on y and solving we get $y(t)$ and then substituting that value in second equation (not in first) of (6) we will obtain the solution for x.
2. Since here we are required two exactly arbitrary constants, therefore finding general solution of x from the differential equation (7), we substitute that solution in first equation of (6), not in the second equation of (6).
3. We should remember that D is an operator. Therefore the differential equation (7) can not be obtained by multiplying $(D + b_2)$ in the first equation of (6), b_1 by second equation of (6) and then subtracting. Instead here we perform and operation which is operating first equation of (6) by the operator $(D + b_2)$ and operating second equation of (6) by b_1 and then performing subtraction we obtain the second order differential equation of the form (7).

- **Method-2 :** In this method, at first we would differentiate any one equation of (6) with respect to t. Without loss of generality let us differentiate first equation of (6) and using the operator $D = \frac{d}{dt}$, we have, $D^2x + a_1Dx + b_1Dy = 0$.

i.e., $D^2x + a_1Dx + b_1[-a_2x + b_2y] = 0$ [From second equation (6)]

i.e., $D^2x + a_1Dx - a_2b_1x - b_1b_2y = 0$

i.e., $D^2x + a_1Dx - a_2b_1x + b_2(D + a_1)x = 0$ [From first equation of (6)]

i.e., $D^2x + (a_1 + b_2)Dx + (a_1b_2 - a_2b_1)x = 0$ ...(8)

Now the differential equation (8) has same form with the equation (7) of previous method (Method-1). Using the method described in that chapter, we may get the general solution for $x = x(t)$, which contains two arbitrary constants. Next substituting that solution $x = x(t)$ in first equation of (6), we obtain,

$$y(t) = -\frac{1}{b_1}[a_1x + x'(t)]$$

and combining we have required general solution.

Notes

1. Like previous method, here also we may differentiate the second equation of (6) and then we get general solution of y which on using again in second equation of (6), the general solution of x can be obtained.
2. It is easy to derive that both method-1 and method-2, would give identical solution since equation (7) and equation (8) are identical to each other.

- **Method-3 : Euler's method for solving the system of homogeneous linear differential equations.**

To solve the system (6) [or system (4)], we recall the differential equation (6) as b

$$\left.\begin{aligned}(D + a_1)x + b_1y &= 0\\ a_2x + (D + b_2)y &= 0\end{aligned}\right\}$$

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Let $x(t) = c_1e^{\lambda t}$ and $y(t) = c_2e^{\lambda t}$ be two trial solutions of system (6) with $c_1 \neq 0$, $c_2 \neq 0$ and λ be a constant.

Thus we have, $Dx = c_1\lambda e^{\lambda t}$ and $Dy = c_2\lambda e^{\lambda t}$ and we have from (6),

$$[(\lambda + a_1)c_1 + b_1c_2]e^{\lambda t} = 0$$

$$[a_2c_1 + (\lambda + b_2)c_2]e^{\lambda t} = 0$$

i.e., $\left.\begin{array}{l}(\lambda + a_1)c_1 + b_1c_2 = 0\\ a_2c_1 + (\lambda + b_2)c_2 = 0\end{array}\right\}$ ... (9)

For non-trivial solutions of c_1 and c_2 we must have from (9)

$$\begin{vmatrix}\lambda + a_1 & b_1\\ a_2 & \lambda + b_2\end{vmatrix} = 0$$

i.e., $(\lambda + a_1)(\lambda + b_2) - a_2b_1 = 0$

i.e., $\lambda^2 + (a_1 + b_2)\lambda + (a_1b_2 - a_2b_1) = 0$ ... (10)

Equation (10) is a quadratic equation on λ and we always have two solutions of λ. Let $\lambda = \lambda_1, \lambda_2$ be two solutions obtained from (10). Then we have three possibilities, namely (i) both of λ_1, λ_2 may be real and unequal, (ii) both of λ_1 and λ_2 are real and equal and (iii) λ_1 and λ_2 are complex and conjugate. We discuss all the cases one-by-one below :

- **Case-1** : Let λ_1 and λ_2 both real and unequal.
  In this case $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ be two linearly independent solution for both $x(t)$ and $y(t)$. Therefore, let us agree to write to general solution of x as follows :

  $x(t) = A\,e^{\lambda_1 t} + B\,e^{\lambda_2 t}$; where A and B are arbitrary constants. ...(11)

  Then using (11) in the first equation of system (6) we can obtain $y(t)$ and combining we have the required general solution.

- **Case-2** : Let λ_1 and λ_2 both real and equal.
  In this case $e^{\lambda_1 t}$ (or $e^{\lambda_2 t}$) and $t\,e^{\lambda_1 t}$ are two independent solutions for both the variables x and y. Therefore, again let us agree to write the general solution of x as follows :

  $x(t) = (A + Bt)\,e^{\lambda_1 t}$ ...(12)

  where A and B are two arbitrary constants.
  Using (12) in the first equation of system (6) we may obtain $y(t)$ and combining we have the required general solution.

- **Case-3** : Let λ_1 and λ_2 we complex conjugate, such that $\lambda_1, \lambda_2 = \alpha \pm i\beta$.
  In this case the general solution of x cum be written as follows :

  $x(t) = e^{\alpha t}\,(A\cos\beta t + B\sin\beta t)$ ...(13)

  where A and B are two arbitrary constants.
  With the help of the solution of $x(t)$ from (13), we have the general solution of $y(t)$ from the first equation of system (6) which also contain two arbitrary constants A and B.

Note

Among the above mentioned three methods, any method can be used to solve the system of differential equation (6) [or the system (4)] although the first two methods are almost equivalent.

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- **Method for solving non-homogeneous simultaneous linear differential equa-tion (5) [or system (3)] :**

Among various methods, below we use the most simple method to evaluate the system of differential equation (5) [or system (3)]. First we differentiate the first equation of (5) with respect to t and using the operator $D \equiv \frac{d}{dt}$, we have,

$$D^2x + a_1Dx + b_1Dy = \phi_1'(t)$$

i.e., $D^2x + a_1Dx + b_1[\phi_2(t) - a_2x - b_2y] = \phi_1'(t)$ [using second equation of (5)]

i.e., $D^2x + a_1Dx - b_1a_2x - b_2(b_1y) = \phi_1'(t) - b_1\phi_2(t)$

i.e., $D^2x + a_1Dx - b_1a_2x - b_2[\phi_1(t) - (D + a_1)x] = \phi_1'(t) - b_1\phi_2(t)$

[using first equation of (5)]

i.e., $D^2x + (a_1 + a_2)x + (a_1b_2 - b_1a_2)x = \phi_1'(t) - b_1\phi_2(t) + b_2\phi_1(t)$

If we write $\phi(t) = \phi_1'(t) - b_1\phi_2(t) + b_2\phi_1(t)$, then from above we have,

$$D^2x + (a_1 + b_2)x + (a_1b_2 - b_1a_2)x = \phi(t) \qquad ...(14)$$

Equation (14) is a non-homogeneous second order differential equation with constant co-efficients. Hence, its general solution can be obtained by combining the complementary func-tion (5) which is the general solution of differential equation (7) [or equivalently (8)] and the particular integral (P.I.). Clearly the C.F. contains two arbitrary constants. Substituting this solution in first equation of system (5) we obtain the general solution for y and hence com-bining we get the required general solution for the system of non-homogeneous differential equation (5) [or equivalently (3)]

Note

> If the system of differential equation contains more than two equation and two dependent variables, then the generalized from of above mentioned method can be used to obtain required general solution which will be explained in subsequent examples.

11.2 Solution Procedure of Type-II Differential Equation

Let us consider a set of two differential equations containing three variables x, y, z as follows :

$$\left.\begin{aligned} f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz = 0 \\ \text{and} \quad \phi_1(x, y, z)dx + \phi_2(x, y, z)dy + \phi_3(x, y, z)dz = 0 \end{aligned}\right\} = 0 \qquad ... (15)$$

The equivalent form of (15) can be obtained from cross-multiplication of the system of equation (15), which is

$$\frac{dx}{f_2\phi_3 - \phi_2f_3} = \frac{dy}{f_3\phi_1 - f_1\phi_3} = \frac{dz}{f_1\phi_2 - f_2\phi_1} \quad i.e., \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad ...(16)$$

where P, Q and R all are functions of x, y, z.

In this chapter we will describe the solution procedure of the system of simultaneous dif-ferential equation (16). Although we have already derived the simultaneous equations (15) and (16) are in same form, still for examination purpose we mainly interested on solving the simultaneous equation of the form (16). Below we describe the rules or method to solve the

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quation of the form (16). Further it should be noted that since there are three first order independent differential term therefore the general solution of (16) should contain two arbitrary conatants.

- **Rule-1** : If the system (16) in such a from that, any two fractions after considering them separately they can be solved and that solution contains one arbitrary constant. Similarly considering another two fractions we may obtain another set of general solution having another constant. Thus finally those two solutions together represent the general solution of the given system of differential equation (16).

Illustration-1 : Solve : $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Considering, the first two fractions, we have,

$\frac{dx}{yz} = \frac{dy}{zx}$ giving $x\,dx = y\,dy$ [since $z \neq 0$]

Integrating we have, $x^2 - y^2 = A$, where A is an arbitrary constant.

Again considering last two fractions, we have

$\frac{dy}{zx} = \frac{dz}{xy}$ giving $y\,dy = z\,dz$ [since $x \neq 0$]

Integrating we have, $y^2 - z^2 = B$, where B is an arbitrary constant.

Hence the required general solution can be given by

$x^2 - y^2 = A$ and $y^2 - z^2 = B$.

- **Rule-2** : Among the three fractions of system of differential equations (16), if it can be obtained a general solution from any two fractions of (16) but another general solution can not obtained from direct computations of other two fractions. But using the constant value obtained from first general solution, we can obtained another solution containing another arbitrary constant, then those solutions together represent the general solution of the system of differential equation (16).

Illustration-2 : Solve : $\frac{dx}{3} = \frac{dy}{-1} = \frac{dz}{z\sin(x+3y)}$

Considering first two fractions, we have, $\frac{dx}{3} = \frac{dy}{-1}$.

i.e., $dx + 3dy = 0$ an on integration we have,

$x + 3y = A$, A being an arbitrary constant.

Again considering second and third fraction we have,

$\frac{dy}{-1} = \frac{dz}{z\sin(x+3y)}$ giving $dy + \frac{dz}{z\sin A} = 0$, since $A = x + 3y$.

i.e., $\sin A\, dy + \frac{dz}{z} = 0$.

Integrating we get, $\log z + (\sin A)y = B$, where B is an arbitrary constant.

i.e., $\log z + y\sin(x + 3y) = B$, since $A = x + 3y$.

Hence the required general solution is given by

$x + 3y = A$ and $\log z + y\sin(x + 3y) = B$.

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- **Rule-3 :** Let $\Psi_1(x, y, z)$, $\Psi_2(x, y, z)$ and $\Psi_3(x, y, z)$ be three suitable functions of x, y and z or constants such that the system of differential equation (16) can be re-written as follows :

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} = \frac{\Psi_1 dx+\Psi_2 dy+\Psi_3 dz}{\Psi_1 P+\Psi_2 Q+\Psi_3 R} \quad ...(17)$$

in such a way that $\Psi_1 P + \Psi_2 Q + \Psi_3 R = 0$ then we have from (17), $\Psi_1 dx + \Psi_2 dy + \Psi_3 dz = 0$ and on integrating we have, $\phi(x, y, z) = A$, A being an arbitrary constant.

To find another solution it is required to find another set of three functions $\eta_1(x, y, z)$, $\eta_2(x, y, z)$ and $\eta_3(x, y, z)$ such that $\eta_1 P + \eta_2 Q + \eta_3 R = 0$ and then we have $\eta_1 dx + \eta_2 dy + \eta_3 dz = 0$ which on integration gives, $\Psi(x, y, z) = B$; B being an arbitrary constant.

Hence the required general solution is $\phi(x, y, z) = A$ and $\Psi(x, y, z) = B$.

- **Illustration-3 : Solve :** $\frac{dx}{yz}=\frac{dy}{-zx}=\frac{3dz}{xy}$

We have, $\frac{dx}{yz}=\frac{dy}{-zx} = \frac{3dz}{xy}=\frac{x\,dx+2y\,dy+3z\,dz}{xyz-2xyz+xyz} = \frac{x\,dx+2y\,dy+3z\,dz}{0}$

i.e., $xdx + 2ydy + 2zdz = 0$

On integration we have, $\frac{x^2}{2}+y^2+\frac{3z^2}{2}=\frac{A}{2}$

i.e., $x^2 + 2y^2 + 3z^2 = A$, A being an arbitrary constant.

Again considering the factors as $(-x)$, $(-4y)$ and $(-3, z)$ we have

$$\frac{dx}{yz}=\frac{dy}{-zx}=\frac{3\,dz}{xy}=\frac{-x\,dx-4y\,dy-9z\,dz}{-xyz+4xyz-3xyz}=\frac{-x\,dx-4y\,dy-9z\,dz}{0}$$

i.e., $xdx + 4ydy + 9zdz = 0$

Integrating we get, $x^2 + 4y^2 + 9z^2 = B$, B being an arbitrary constant.

Hence the required general solution is

$$x^2 + 2y^2 + 3z^2 = A \text{ and } x^2 + 4y^2 + 9z^2 = B.$$

- **Rule-4 :** Like rule-3, if these functions Ψ_1, Ψ_2 and Ψ_3 can be found such that from (16) we can write,

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} = \frac{\Psi_1 dx+\Psi_2 dy+\Psi_3 dz}{\Psi_1 P+\Psi_2 Q+\Psi_3 R} = \frac{d(\Psi_1 P+\Psi_2 Q+\Psi_3 R)}{\Psi_1 P+\Psi_2 Q+\Psi_3 R} \quad ...(18)$$

Then combining fifth ratio with either of first three ratio or some combination of first three ratios give a relation which can be integrated. Then after integrating, we have $\phi(x, y, z) = A$ as one solution. Similarly combining fifth ratio of (18) along with other fraction among first three of (18) or any combination among the ratios from first three and fifth may give $\Psi(x, y, z) = B$. Then $\phi(x, y, z) = A$ and $\Psi(x, y, z) = B$ [where A and B are arbitrary constants] together gives required general solution.

- **Illustration-4 : Solve :** $\frac{dx}{y-2}=\frac{dy}{x+2}=\frac{dz}{z}$

The given system can be written as :

$$\frac{dx}{y-2}=\frac{dy}{x+2}=\frac{dz}{z} = \frac{dx+dy+dz}{(y-2)+(x+2)+z} = \frac{d(x+y+z)}{x+y+z}$$

Considering third and fifth factors we have,

$\frac{dz}{z} = \frac{d(x+y+z)}{x+y+z}$, on integrating we have, $\log z + \log A = \log(x + y + z)$

i.e., $x + y + z = Az$

Again considering first two fractions we have,

$\frac{dx}{y-2} = \frac{dy}{x+2}$ giving, $xdx + 2dx = ydy - 2dy$

On integration we have, $\frac{x^2}{2} + 2x = \frac{y^2}{2} - 2y + \frac{B}{2}$

i.e., $x^2 - y^2 + 4(x + y) = B$ *i.e.*, $(x + y)(x - y + 4) = 0$

Thus the required general solution is

$x + y + z = Az$ and $(x + y)(x - y + 4) = B$, where A and B are two arbitrary constants.

On the basis of above mentioned four rules, below we provide some worked out examples to solve simultaneous differential equation of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Illustrative Examples

1. **Suppose $\frac{dx}{dt} - 7x + y = 0$, $\frac{dy}{dt} - 2x - 5y = 0$. Express x in terms of t.**

Solution Given $\frac{dx}{dt} - 7x + y = 0$ ...(1)

$\frac{dy}{dt} - 2x - 5y = 0$ ...(2)

From (1) we get $y = -\frac{dy}{dt} + 7x$

Now from (2) we get, $\frac{d}{dt}\left[-\frac{dx}{dt} + 7x\right] - 2x + 5\frac{dx}{dt} - 35x = 0$

i.e., $-\frac{d^2x}{dt^2} + 7\frac{dx}{dt} - 2x + 5\frac{dx}{dt} - 35x = 0 \Rightarrow \frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 37x = 0$ ... (3)

Let $x = ce^{mt}$ be a trial solution of (3) with $c \neq 0$

Then the auxiliary equation is $m^2 - 12m + 37 = 0$ [since $e^{mt} \neq 0$]

giving, $m = \frac{12 \pm \sqrt{144 - 148}}{2} = \frac{12 \pm 2i}{2} = 6 \pm i$

Hence the required expression of x is $e^{6t}[A\cos t + B\sin t]$.

2. **Suppose $\frac{dx}{dt} - 5x - 4y = 0$, $\frac{dy}{dt} + x - y = 0$. Express y in form of t.**

Solution $\frac{dx}{dt} - 5x - 4y = 0$ ... (1)

$\frac{dy}{dt} + x - y = 0$ ... (2)

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From (2), we get, $x = -\frac{dy}{dt} + y$

Now from (1) we get,

$$\frac{d}{dt}\left(-\frac{dy}{dt} + y\right) + 5\frac{dy}{dt} - 5y - 4y = 0 \Rightarrow -\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\frac{dy}{dt} - 9y = 0 \Rightarrow \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0$$

Let $y = ce^{mt}$ be a trial solution, where $c \neq 0$

Thus the auxiliary equation is

$m^2 - 6m + 9 = 0$ [since $e^{mt} \neq 0$]

i.e., $(m-3)^2 = 0$ giving $m = 3, 3$

Therefore $y = (A + Bt)e^{3t}$, where A and B are two arbitrary constants.

3. **Solve :** $\frac{dx}{dt} + 3x - 4y = 0,\ \frac{dy}{dt} + 2x - 3y = 0$

Solution $\frac{dx}{dt} + 3x - 4y = 0$ ...(1)

$\frac{dy}{dt} + 2x - 3y = 0$ ...(2)

From (1) we get, $y = \frac{1}{4}\left(\frac{dx}{dt} + 3x\right)$ ...(3)

Now, from (2) we get, $\frac{1}{4}\frac{d}{dt}\left(\frac{dx}{dt} + 3x\right) + 2x - \frac{3}{4}\left(\frac{dx}{dt} + 3x\right) = 0$

i.e., $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 8x - 3\frac{dx}{dt} - 9x = 0 \Rightarrow \frac{d^2x}{dt^2} - x = 0$

i.e., $(D^2 - 1)x = 0$, where $D \equiv \frac{d}{dt}$

Let $x = ce^{mt}$ be a trial solution, with $c \neq 0$. ... (4)

Thus the auxiliary equation is $m^2 - 1 = 0$ giving $m = \pm 1$ [since $e^{mt} \neq 0$]

So, $x = Ae^t + Be^{-t}$, where A and B are two arbitrary constants.

Therefore $\frac{dx}{dt} = Ae^t - Be^{-t}$

Form (3), we get, $y = \frac{1}{4}[Ae^t - Be^{-t} + 3Ae^t + 3Be^{-t}]$

Therefore $y = \frac{1}{4}[4Ae^t + 2Be^{-t}] = Ae^t + \frac{B}{2}e^{-t}$

Hence the required solution is $x = Ae^t + Be^{-t},\ y = Ae^t + \frac{B}{2}e^{-t}$

4. **Solve :** $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0,\ \frac{dy}{dt} + 5x + 3y = 0$

[CH 1992, 1994]

Solution Introducing the operator $D \equiv \frac{d}{dt}$, the given system of differential equations can be written as :

$(D + 2)x + (D + 1)y = 0$

$5x + (D + 3)y = 0$ ...(1)

- **Method-1** : Eliminating x between (1) and (2), we have

$[5(D + 1) - (D + 2)(D + 3)]y = 0$

i.e., $-(D^2 + 1)y = 0$ *i.e.*, $(D^2 + 1)y = 0$ ...(3)

Let $y = ce^{mt}$ $(c \neq 0)$ be a trial solution of (3).

Then A.E. becomes $m^2 + 1 = 0$ giving $m = \pm i$.

Thus the general solution of (3) becomes as

$y = A\cos t + B\sin t$, where A and B are two arbitrary constants. ...(4)

Now from (4) we have, $\frac{dy}{dt} = -A\sin t + B\cos t$

From (2) we have,

$5x + (-A\sin t + B\cos t) + 3(A\cos t + B\sin t) = 0$

i.e., $5x = (A - 3B)\sin t - (3A + B)\cos t$

i.e., $x = \frac{1}{5}(A - 3B)\sin t - \frac{1}{5}(3A + B)\cos t$ ...(5)

Hence from (4) and (5) the general solution of given differential equation is

$x(t) = \frac{1}{5}(A - 3B)\sin t - \frac{1}{5}(3A + B)\cos t$ and $y(t) = A\cos t + B\sin t$,

where A and B are two arbitrary constants.

Note:

Instead of eliminating x between (1) and (2), if we can eliminate y between (1) and (2) then also general solution can be obtained.

- **Method-2** : Introducing the operator $D \equiv \frac{d}{dt}$, the given system can be written as

$(D + 2)x + (D + 1)y = 0$ ... (1)

$5x + (D + 3)y = 0$ ... (2)

Differentiating (2) with respect to t we have

$5Dx + (D^2 + 3D)y = 0$ ... (3)

Now multiplying (1) by 5 and (2) by 2 and then subtracting we have,

$5Dx + 5(D + 1)y - 2(D + 3)y = 0$

i.e., $5Dx + (3D - 1)y = 0$ ... (4)

Now, subtracting (4) from (3) we have, $(D^2 + 1)y = 0$ ... (5)

Clearly equation (5) has same form with the equation (3) of **method-1**.

So proceeding in same way, we may get the general solution as follows :

$y(t) = A\cos t + B\sin t$

and $x(t) = \frac{1}{5}(A - 3B)\sin t - (3A + B)\cos t$, where A and B are two arbitrary constants.

5. Solve : $\frac{dx}{dt} - 7x + y = 0$, $\frac{dy}{dt} - 2x - 5y = 0$ **[BH 1991; KH 2003]**

Solution

- **Method-1** : Here the given system of differential equation is :

$\frac{dx}{dt} - 7x + y = 0$ ...(1)

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$$\frac{dy}{dt}-2x-5y=0 \quad \text{...(2)}$$

Introducing the operator $D \equiv \frac{d}{dt}$, the above system of equations can be re-written as :

$$(D-7)x+y=0 \quad \text{...(3)}$$

$$-2x+(D-5)y=0 \quad \text{...(4)}$$

Eliminating y between (3) and (4) we have,

$$(D-5)(D-7)x+2x=0 \text{ i.e., } (D^2-12D+37)x=0 \quad \text{...(5)}$$

Let $x = ce^{mt}$ $(c \neq 0)$ be a trial solution of (5). Then the auxiliary equation of (5) becomes :

$$m^2-12m+37=0 \text{ giving } m=6 \pm i.$$

Hence the general solution can be written from (5) as :

$$x(t) = e^{6t}(A\cos t + B\sin t) \quad \text{...(6)}$$

where A and B are two arbitrary constants.

Differentiating (6) with respect to t we have,

$$\frac{dx}{dt} = e^{6t}(-A\sin t + B\cos t) + 6e^{6t}(A\cos t + B\sin t)$$

$$= e^{6t}\{(6A+B)\cos t + (6B-A)\sin t\}$$

Using the values of x and $\frac{dx}{dt}$ in (1), we have,

$$y = 7x - \frac{dx}{dt} = 7e^{6t}(A\cos t + B\sin t) - e^{6t}\{(6A+B)\cos t + (6B-A)\sin t\}$$

$$\text{i.e., } y(t) = e^{6t}\{(A-B)\cos t + (A+B)\sin t\} \quad \text{... (7)}$$

Thus (6) and (7) together gives the general solutions.

- **Method-2 :** The given system of differential equations can be written as :

$$\frac{dx}{dt}-7x+y=0 \quad \text{...(1)}$$

$$\frac{dy}{dt}-2x-5y=0 \quad \text{...(2)}$$

Differentiating (1) once with respect to t, we have,

$$\frac{d^2x}{dt^2}-7\frac{dx}{dt}+\frac{dy}{dt}=0$$

i.e., $\frac{d^2x}{dt^2}-7\frac{dx}{dt}+(2x+5y)=0$ [From (2)]

i.e., $\frac{d^2x}{dt^2}-7\frac{dx}{dt}+2x+5\left(7x-\frac{dx}{dt}\right)=0$ [From (1)]

i.e., $\frac{d^2x}{dt^2}-12\frac{dx}{dt}+37x=0$ ...(3)

Equation (3) has same form of the equation (5) of **method-1**.

So proceeding same way as **method-1**, the general solution of the given system of differential equation can be written as :

$$x(t) = e^{6t}(A\cos t + B\sin t)$$

and $y(t) = e^{6t}\{(A-B)\cos t + (A+B)\sin t\}$

where A and B are two arbitrary constants.

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Note

The problem can be solved by either eliminating x between two differential equation or finding a second order differential equation on y.

- **Method-3 :** The given system of differential equation can be written as :

$$\left.\begin{aligned} \frac{dx}{dt} &= 7x - y \\ \frac{dy}{dt} &= 2x + 5y \end{aligned}\right\} \qquad \ldots (1)$$

The system (1) is equivalent to the system

$$\frac{dx}{dt} = PX \text{, where } X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } P = \begin{pmatrix} 7 & -4 \\ 2 & 5 \end{pmatrix} \qquad \ldots (2)$$

Let the trial solution of the system (1) be $x = a e^{\lambda t}$, and $y = b e^{\lambda t}$ where a, b are constants such that $a \neq 0$, $b \neq 0$.

Then we have from (1),

$$\left.\begin{aligned} a\lambda e^{\lambda t} &= 7a e^{\lambda t} - b e^{\lambda t} \\ b\lambda e^{\lambda t} &= 2a e^{\lambda t} + 5b e^{\lambda t} \end{aligned}\right\} \qquad \ldots (3)$$

From (3) we have

$$\left.\begin{aligned} (\lambda - 7)a + b &= 0 \\ -2a + (\lambda - 5)b &= 0 \end{aligned}\right\} \qquad \ldots (4)$$

Form non-trivial values of a and b, we have from (4),

$$\begin{vmatrix} \lambda - 7 & 1 \\ -2 & \lambda - 5 \end{vmatrix} = 0$$

i.e., $(\lambda - 7)(\lambda - 5) + 2 = 0$ giving $\lambda^2 - 12\lambda + 37 = 0$

i.e., $\lambda = 6 \pm i$.

Therefore the required general solution is

$x(t) = e^{6t}(A\cos t + B\sin t)$, where A and B are two arbitrary constants.

Differentiating we have,

$$\begin{aligned} \frac{dx}{dt} &= 6e^{6t}(A\cos t + B\sin t) + e^{6t}(-A\sin t + B\cos t) \\ &= (6A + B)e^{6t}\cos t + (6B - A)e^{6t}\sin t \end{aligned}$$

From the first equation of (1) we have,

$$\begin{aligned} y = 7x - \frac{dx}{dt} &= 7e^{6t}(A\cos t + B\sin t) - (6A + B)e^{6t}\cos t - (6B - A)e^{6t}\sin t \\ &= e^{6t}[(A - B)\cos t + (A + B)\sin t] \end{aligned}$$

Hence the required solution gives,

$x(t) = e^{6t}(A\cos t + B\sin t)$

and $y(t) = e^{6t}[(A - B)\cos t + (A + B)\sin t]$, where A and B are two arbitrary constants.

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6. Solve : $\frac{dx}{dt}+4x+3y=t,\ \frac{dx}{dt}+2x+5y=e^t$ [CH 2005; NBH 2000, 20..]

Solution Here the given system of differential equations is

$$\frac{dx}{dt}+4x+3y=t \quad ...(1)$$

$$\frac{dx}{dt}+2x+5y=e^t \quad ...(2)$$

Differentiating (1) with respect to t we have,

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3\frac{dy}{dt}=1 \quad i.e., \quad \frac{d^2x}{dt^2}+4\frac{dx}{dt}+3(e^t-2x-5y)=1 \quad \text{[From (2)]}$$

i.e., $\frac{d^2x}{dt^2}+4\frac{dx}{dt}-6x-15y=1-3e^t$ i.e., $\frac{d^2x}{dt^2}+4\frac{dx}{dt}-6x-\frac{15}{3}\left(t-\frac{dx}{dt}-4x\right)=1-3e^t$ [From (1)]

i.e., $\frac{d^2x}{dt^2}+9\frac{dx}{dt}+14x=-3e^t+5t+1$

i.e., $(D^2+9D+14)x=-3e^t+5t+1$, where $D\equiv\frac{d}{dx}$ ...(3)

The reduced equation of (3) is $(D^2+9D+14)x=0$ ...(4)

Let $x(t)=ce^{mt}(c\neq 0)$ be a trial solution of (4). Then the A.E. of (4) becomes :

$$m^2+9m+14=0 \quad i.e., \quad (m+2)(m+7)=0 \text{ giving } m=-2,-7.$$

Thus the C.F. of (3) is $x_c=Ae^{-2t}+Be^{-7t}$, where A and B are arbitrary constants.

Next if $x=x_p$ be a P.I. of (3), then we have from (3),

$$x_p=\frac{1}{D^2+9D+14}(-3e^t+5t+1)=\frac{1}{D^2+9D+14}(-3e^t)+\frac{1}{D^2+9D+14}(5t+1)$$

$$=-\frac{3}{24}e^t+\frac{1}{14}\left(1+\frac{9D+D^2}{14}\right)^{-1}(1+5t)=-\frac{1}{8}e^t+\frac{1}{14}\left(1-\frac{9D+D^2}{14}+\dots\right)(1+5t)$$

$$=-\frac{1}{8}e^t+\frac{1}{14}\left(1+5t-\frac{45}{14}\right)=-\frac{1}{8}e^t+\frac{5}{14}t-\frac{31}{196}$$

Thus the general solution for $x(t)$ is

$$x(t)=x_c+x_p=Ae^{-2t}+Be^{-7t}-\frac{1}{8}e^t+\frac{5}{14}t-\frac{31}{196} \quad ...(5)$$

Now differentiating (5) with respect to t we get,

$$\frac{dx}{dt}=-2Ae^{-2t}-7Be^{-7t}-\frac{1}{8}e^t+\frac{5}{14} \quad ...(6)$$

Using (5) and (6) in (1) we have,

$$y(t)=\frac{1}{3}\left[t-4x-\frac{dx}{dt}\right]=\frac{1}{3}\left[t-4\left\{Ae^{-2t}+Be^{-7t}-\frac{1}{8}e^t+\frac{5}{14}t-\frac{31}{196}\right\}-\left\{-2Ae^{-2t}-7Be^{-7t}-\frac{1}{8}e^t+\frac{5}{14}\right\}\right]$$

$$=-\frac{2}{3}Ae^{-2t}+Be^{-7t}+\frac{5}{24}e^t-\frac{1}{7}t+\frac{9}{98} \quad ...(7)$$

Thus (5) and (7) together gives the general solution of the given system of differential equation.

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7. Solve : $\frac{dx}{dt}+3x+y=e^t,\ \frac{dy}{dt}-x+y=e^{2t}$

Solution Given $\frac{dx}{dt}+3x+y=e^t$ ...(1)

$\frac{dy}{dt}-x+y=e^{2t}$ ...(2)

From (1) we get, $y = e^t - 3x - \frac{dx}{dt}$ ...(3)

Now form (2) we get,

$$\frac{d}{dt}\left(e^t-3x-\frac{dx}{dt}\right)-x+\left(e^t-3x-\frac{dx}{dt}\right)=e^{2t} \Rightarrow e^t-3\frac{dx}{dt}-\frac{d^2x}{dt^2}-x+e^t-3x-\frac{dx}{dt}=e^{2t}$$

i.e., $-\frac{d^2x}{dt^2}-4\frac{dx}{dt}-4x=e^{2t}-2e^t \Rightarrow \frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=2e^t-e^{2t}$

i.e., $(D^2+4D+4)x = 2e^t - e^{2t}$, where $D \equiv \frac{d}{dt}$ ...(4)

Let $x = ce^{mt}$ be a trial solution of $(D^2 + 4D + 4)x = 0$, where $c \neq 0$.

So, the auxiliary equation is $m^2 + 4m + 4 = 0$ [since $e^{mt} \neq 0$]

i.e., $(m + 2)^2 = 0$ giving $m = -2, -2$

Thus the C.F. = $(A + Bt)\, e^{-2t}$, where A and B are two arbitrary constants.

Now P.I. $= \frac{1}{(D+2)^2}(2e^t-e^{2t}) = 2\cdot\frac{1}{(D+2)^2}\cdot e^t-\frac{1}{(D+2)^2}\cdot e^{2t}$

$= \frac{2e^t}{(1+2)^2}-\frac{e^{2t}}{(2+2)^2} = \frac{2}{9}e^t-\frac{1}{16}e^{2t}$

The solution for x is $x = (A + Bt)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$

Now, from (3) we get

$$y = e^t-3(A+Bt)e^{-2t}-\frac{2}{3}e^t+\frac{3}{16}e^{2t}-\frac{d}{dt}\left[(A+Bt)e^{-2t}+\frac{2}{9}e^t-\frac{1}{16}e^{2t}\right]$$

$$= e^t-3(A+Bt)e^{-2t}-\frac{2}{3}e^t+\frac{3}{16}e^{2t}-\left[(A+Bt)(-2e^{-2t})+e^{-2t}\cdot B+\frac{2}{9}e^t-\frac{1}{8}e^{2t}\right]$$

So, $y = (-A + Bt)e^{-2t} - Be^{-2t} + \frac{1}{9}e^t - \frac{5}{16}e^{2t}$.

Therefore the required complete solution is

$x = (A + Bt)e^{-2t} + \frac{1}{9}e^t - \frac{1}{16}e^{2t}$ and $y = (-A + Bt)e^{-2t} - Be^{-2t} + \frac{1}{9}e^t - \frac{5}{16}e^{2t}$.

8. Solve : $\frac{dx}{dt}+5x+y=e^t,\ \frac{dx}{dt}-x+3y=e^{2t}$ [VH 2009]

Solution Here the given system of differential equations is

$\frac{dx}{dt}+5x+y=e^t$ ... (1)

$\frac{dy}{dt}-x+3y=e^{2t}$ ... (2)

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Differentiating (1) with respect to t we have,

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+\frac{dy}{dt}=e^t$$

i.e., $$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+(e^{2t}+x-3y)=e^t \quad \text{[From (2)]}$$

i.e., $$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+x+e^{2t}-3\left(e^t-\frac{dx}{dt}-5x\right)=e^t \quad \text{[From (1)]}$$

i.e., $$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=4e^{2t}-e^{2t} \qquad ...(3)$$

The reduced equation of (3) becomes as $\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=0$ ...(4)

Let the trial solution of (4) be $x = ce^{mt}$ $(c \neq 0)$. Then the auxiliary equation of (4) is

$m^2 + 8m + 16 = 0$ giving $(m + 4)^2 = 0$ i.e., $m = -4, -4$.

Thus the C.F. of (3) is $x_c = (A + Bt)\,e^{-4t}$, where A and B are two arbitrary constants.

To find P.I. of (3), let $x = x_p$ be a P.I. Now introducing $D \equiv \frac{d}{dt}$, we have from (3),

$$x_p = \frac{1}{D^2+8D+16}(4e^t-e^{2t}) = \frac{1}{(D+4)^2}(4e^t)-\frac{1}{(D+4)^2}(e^{2t})$$

$$= 4\cdot\frac{1}{(1+4)^2}e^t-\frac{1}{(2+4)^2}e^{2t} \quad \left[\text{since } \frac{1}{f(D)}e^{at}=\frac{1}{f(a)}e^{at}\right]$$

$$= \frac{4}{25}e^t-\frac{1}{36}e^{2t}$$

So, the general solution of (3) is $x = x_c + x_p$

i.e., $$x(t) = (A+Bt)e^{-4t}+\frac{4}{25}e^t-\frac{1}{36}e^{2t} \qquad ...(5)$$

Now, differentiating (5) with respect to t we have,

$$\frac{dx}{dt} = \frac{4}{25}e^t-\frac{1}{18}e^{2t}-4Ae^{-4t}+Be^{-4t}-4Bte^{-4t}$$

Thus from (1) we have,

$$y(t) = e^t-\frac{dx}{dt}-5x(t)$$

$$= e^t-\left(\frac{4}{25}e^t-\frac{1}{18}e^{2t}-4Ae^{-4t}+Be^{-4t}-4Bte^{-4t}\right)-5\left\{(A+Bt)e^{-4t}+\frac{4}{25}e^t-\frac{1}{36}e^{2t}\right\}$$

$$= \left(1-\frac{4}{25}-\frac{20}{25}\right)e^t+\left(\frac{1}{18}+\frac{5}{36}\right)e^{2t}+(4A-B+4Bt-5A-5Bt)e^{-4t}$$

$$= -(A+6B+Bt)e^{-4t}+\frac{1}{25}e^t+\frac{7}{36}e^{2t} \qquad ...(6)$$

Thus from (5) and (6), the general solution of the given system of differential equations is

$$x(t) = (A+Bt)e^{-4t}+\frac{4}{25}e^t-\frac{1}{36}e^{2t}$$

$$y(t) = -(A+6B+Bt)e^{-4t}+\frac{1}{25}e^t+\frac{7}{36}e^{2t}$$

where A and B are two arbitrary constants.

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9. Solve : $\frac{dx}{dt} = 5x + 4y$, $\frac{dy}{dt} = -x + y$ [VU 2008]

Solution Here the given system of differential equations is

$$\frac{dx}{dt} = 5x + 4y \quad \ldots (1)$$

$$\frac{dy}{dt} = -x + y \quad \ldots (2)$$

Differentiating (1) with respect to t we have,

$$\frac{d^2x}{dt^2} = 5\frac{dx}{dt} + 4\frac{dy}{dt} = 5\frac{dx}{dt} + 4(-x+y) \text{ [From (2)]}$$

$$= 5\frac{dx}{dt} - 4x + \left(\frac{dx}{dt} - 5x\right) \text{ [From (1)]}$$

i.e., $$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 9x = 0 \quad \ldots (3)$$

Let $x(t) = ce^{mt}$ $(c \neq 0)$ be a trial solution of (3). Then the auxiliary equation of (3) becomes $m^2 - 6m + 9 = 0$ giving $(m-3)^2 = 0$ *i.e.,* $m = 3, 3$

Thus the general equation of (3) is

$$x(t) = (A + Bt)e^{3t} \quad \ldots (4)$$

where A and B are two arbitrary constants.

Now differentiating (4) with respect to t, we have,

$$\frac{dx}{dt} = 3Ae^{3t} + 3Bte^{3t} + Be^{3t} \quad \ldots (5)$$

Using the values of x and $\frac{dx}{dt}$ in (1), we have,

$$y(t) = \frac{1}{4}\left(\frac{dx}{dt} - 5x\right) = \frac{1}{4}[3Ae^{3t} + 3Bte^{3t} + Be^{3t} - 5Ae^{3t} - 5Bte^{3t}]$$

$$= \frac{1}{4}(B - 2A)e^{3t} - \frac{1}{2}Bte^{3t}$$

Hence the required general solution for the given system is

$$x(t) = (A + Bt)e^{3t}, \quad y(t) = \frac{1}{4}(B - 2A)e^{3t} - Bte^{3t}$$

Alternative Method

Euler's method to solve homogeneous differential equation :

Here the given system of differential equations is

$$\left.\begin{aligned} \frac{dx}{dt} &= 5x + 4y \\ \frac{dy}{dt} &= -x + y \end{aligned}\right\} \quad \ldots (1)$$

Let $x(t) = c_1e^{\lambda t}$ and $y(t) = c_2e^{\lambda t}$ $(c_1, c_2 \neq 0)$ be the trial solution of the system (1). Then using $\frac{dx}{dt} = \lambda c_1 e^{\lambda t}$ and $\frac{dy}{dt} = \lambda c_2 e^{\lambda t}$, we have from (1)

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$$\lambda c_1 e^{\lambda t} = 5c_1 e^{\lambda t} + 4c_2 e^{\lambda t} \quad i.e., (\lambda - 5)c_1 - 4c_2 = 0$$
$$\lambda c_2 e^{\lambda t} = -c_1 e^{\lambda t} + c_2 e^{\lambda t} \quad i.e., c_1 + (\lambda - 1)c_2 = 0 \quad ...(2)$$

For existence of non-trivial solutions of the system (2) we must have, the coefficient determinant should be zero. So from (2), we have,

$$\begin{vmatrix} \lambda - 5 & -4 \\ 1 & \lambda - 1 \end{vmatrix} = 0$$

i.e., $(\lambda - 5)(\lambda - 1) + 4 = 0$ *i.e.*, $\lambda^2 - 6\lambda + 9 = 0$ giving $(\lambda - 3)^2 = 0$

i.e., $\lambda = 3, 3$

Therefore the general solution of x can be written as :

$$x(t) = (A + Bt)e^{3t}$$

where A and B are two arbitrary constants. ...(3)

Now differentiating (2) with respect to t we have,

$$\frac{dx}{dt} = 3Ae^{3t} + 3Bte^{3t} + Be^{3t} \quad ...(4)$$

Using (3) and (4) in first equation of (1), we have,

$$y(t) = \frac{1}{4}\left(\frac{dx}{dt} - 5x\right) = \frac{1}{4}[3Ae^{3t} + 3Bte^{3t} + Be^{3t} - 5\{(A + B)e^{3t}\}] = \frac{1}{4}(B - 2A)e^{3t} - \frac{1}{2}Bte^{3t}$$

Hence required general solution is

$$x(t) = (A + Bt)e^{3t}, \; y(t) = \frac{1}{4}(B - 2A)e^{3t} - \frac{1}{2}Bte^{3t},$$

where A and B are two arbitrary constants.

Note

In this problem e^{3t} and te^{3t} are two independent solutions of both x and y. So the solution of y also can be expressed as $y = (A + Bt)e^{3t}$ and from second equation of (1), we may obtain solution for $x(t)$.

10. Solve : $\frac{dx}{dt} + 3x - y = e^t, \; \frac{dy}{dt} + x + 5y = 0$

[VU 2010]

Solution Here the given system of differential equations is

$$\frac{dx}{dt} + 3x - y = e^t \quad ...(1)$$

$$\frac{dy}{dt} + x + 5y = 0 \quad ...(2)$$

Differentiating (1) with respect to t, we have,

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - \frac{dy}{dt} = e^t \quad i.e., \; \frac{d^2x}{dt^2} + 3\frac{dx}{dt} + (x + 5y) = e^t \quad \text{[From (2)]}$$

i.e., $$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x + 5\left(\frac{dx}{dt} + 3x - e^t\right) = e^t \quad \text{[From (1)]}$$

i.e., $$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 6e^t$$

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The reduced equation of (3) becomes as

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=0 \quad ...(4)$$

Let $x = ce^{mt}$ be a trial solution of (4). Then the auxiliary equation of (4) becomes

$$m^2 + 8m + 16 = 0 \text{ giving } (m + 4)^2 = 0$$

i.e., $m = -4, -4$

Therefore, the C.F. of (3) is $x_c = (A + Bt)e^{-4t}$, where A and B are two arbitrary constants.

Let $x = x_p$ be a P.I. of (3). Then introducing the operator $D \equiv \frac{d}{dt}$, we have from (3),

$$x_p = \frac{1}{D^2+8D+16}(6e^t) = \frac{1}{1+8+16}(6e^t) = \frac{6}{25}e^t \quad \left[\text{since } \frac{1}{f(D)}e^{at} = \frac{1}{f(a)}e^{at}\right]$$

Thus the general solution for x is $x(t) = (A + Bt)e^{-4t} + \frac{6}{25}e^t$

Differentiating we have, $\frac{dx}{dt} = -4Ae^{-4t} - 4Btt^{-4t} + Be^{-4t} + \frac{6}{25}e^t$

From (1) we have,

$$y(t) = \frac{dx}{dt} + 3x - e^t = -4Ae^{-4t} - 4Btt^{-4t} + Be^{-4t} + \frac{6}{25}e^t + 3(A + Bt)e^{-4t} + \frac{18}{25}e^t - e^t$$

i.e., $y(t) = (B - A)e^{-4t} - Bte^{-4t} - \frac{1}{25}e^t$

Therefore, combining we have the required general solution of the given system is as follows :

$$x(t) = (A + Bt)e^{-4t} + \frac{6}{25}e^t$$

and $y(t) = (B - A)e^{-4t} - Bte^{-4t} - \frac{1}{25}e^t$

where A and B are two arbitrary constants.

11. Solve : $\frac{dx}{dt}-\frac{dy}{dt}+2y=\cos 2t,\ \frac{dx}{dt}+\frac{dy}{dt}-2x=\sin 2t$

Solution Given $\frac{dx}{dt}-\frac{dy}{dt}+2y=\cos 2t$ ...(1)

$$\frac{dx}{dt}+\frac{dy}{dt}-2x=\sin 2t \quad ...(2)$$

Adding (1) and (2) we get,

$$2\frac{dx}{dt} + 2y - 2x = \cos 2t + \sin 2t$$

i.e., $y = x - \frac{dx}{dt} + \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t$ ...(3)

From (1), we get,

$$\frac{dx}{dt} - \frac{dx}{dt} + \frac{d^2x}{dt^2} + \sin 2t - \cos 2t + 2x - 2\frac{dx}{dt} + \cos 2t + \sin 2t = \cos 2t$$

i.e., $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = \cos 2t - 2\sin 2t$ ...(4)

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Let $x = ce^{mt}$ be a trial solution of $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0$, where $c \neq 0$

Thus the auxiliary equation is

$$(m^2 - 2m + 2) = 0 \quad [\text{since } e^{mt} \neq 0]$$

giving $m = 1 \pm i$

So, the C.F. $= e^t(A\cos t + B\sin t)$, where A and B are two arbitrary constants.

Now, P.I. $= \frac{1}{D^2 - 2D + 2}(\cos 2t - 2\sin 2t)$

$$= \frac{1}{-2^2 - 2D + 2}(\cos 2t - 2\sin 2t) = -\frac{1}{2}\left[\frac{1}{D+1}(\cos 2t - 2\sin 2t)\right]$$

$$= -\frac{1}{2}\left[\frac{(D-1)}{D^2 - 1}(\cos 2t - 2\sin 2t)\right] = -\frac{1}{2}\left[\frac{D(\cos 2t - 2\sin 2t) - \cos 2t + 2\sin 2t}{-2^2 - 1}\right]$$

$$= -\frac{1}{2}\left[\frac{-2\sin 2t - 4\cos 2t - \cos 2t + 2\sin 2t}{-5}\right] = -\frac{1}{2}\cos 2t$$

Therefore the complete solution of (4) is

$$x = \text{C.F.} + \text{P.I.} \Rightarrow x = e^t(A\cos t + B\sin t) - \frac{1}{2}\cos 2t$$

Now from (3) we get,

$$y = e^t(A\cos t + B\sin t) - \frac{1}{2}\cos 2t - \frac{d}{dt}\left[e^t(A\cos t + B\sin t) - \frac{1}{2}\cos 2t\right] + \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t$$

i.e., $y = e^t(A\cos t + B\sin t) - \frac{1}{2}\cos 2t - [e^t(A\cos t + B\sin t) + e^t(-A\sin t + B\cos t) + \sin 2t]$

$$+\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t$$

i.e., $y = e^t(A\sin t - B\cos t) - \frac{1}{2}\sin 2t$

Therefore the required complete solution is

$$x = e^t(A\cos t + B\sin t) - \frac{1}{2}\cos 2t$$

$$y = e^t(A\sin t - B\cos t) - \frac{1}{2}\sin 2t$$

12. **Solve :** $\frac{dx}{dt} = 3x + 8y,\ \frac{dy}{dt} = -x - 3y$ **with** $x(0) = 6,\ y(0) = -2$.

Solution Given $\frac{dx}{dt} = 3x + 8y$

$$\frac{dy}{dt} = -x - 3y \quad \text{... (1)}$$

From (2) we have, $x = -3y - \frac{dy}{dt}$ ... (2)

So, from (1) we get, ... (3)

$$\frac{d}{dt}\left(-3y - \frac{dy}{dt}\right) = 3\left(-3y - \frac{dy}{dt}\right) + 8y \Rightarrow -3\frac{dy}{dt} - \frac{d^2y}{dt^2} = -9y - 3\frac{dy}{dt} + 8y \Rightarrow \frac{d^2y}{dt^2} - y = 0 \quad \text{... (4)}$$

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Let $y = ce^{mt}$ be a trial solution with $c \neq 0$

So, the auxiliary equation is

$$m^2 - 1 = 0 \quad [\text{since } e^{mt} \neq 0]$$

giving $m = \pm 1$

Thus, $y = Ae^t + Be^t$, where A and B are two arbitrary constants. ...(5)

Now from (3) we get,

$$x = -3(Ae^t + Be^{-t}) - \frac{d}{dt}(Ae^t + Be^{-t})$$

i.e., $x = -3(Ae^t - 2Be^{-t}) - (Ae^t - Be^{-t})$

i.e., $x = -4Ae^t - 2Be^{-t}$ ...(6)

Now accordingly, $x = 6$ and $y = -2$ when $t = 0$

From (5) and (6) we get,

$$-2 = A + B$$

and $6 = -4A - 2B$ giving $2A + B = -3$

Solving we get, $A = -1$, $B = -1$

Therefore the required solution is $\begin{cases} x = 4e^t + 2e^{-t} \\ y = -e^t + e^{-t} \end{cases}$

13. Solve : $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$

Solution Considering first two fractions, we get,

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} \text{ giving } \cot x\,dx = \cot y\,dy$$

Integrating we get,

$\log\sin x = \log\sin y + \log A$, where A is an arbitrary constant.

i.e., $\log\dfrac{\sin x}{\sin y} = \log A$ giving $\dfrac{\sin x}{\sin y} = A$

Again considering first and third fraction we get,

$$\frac{dx}{\tan x} = \frac{dz}{\tan z} \text{ giving } \cot x\,dx = \cot z\,dz$$

Integrating we get, $\log\sin x = \log\sin z + \log B$, where B is an arbitrary constants.

i.e., $\dfrac{\sin x}{\sin z} = B$

So, the general solution is $\dfrac{\sin x}{\sin y} = A$, $\dfrac{\sin x}{\sin z} = B$.

14. Solve : $\dfrac{dx}{yz} = \dfrac{dy}{zx} = \dfrac{dz}{xy}$

Solution Here the given differential equation is

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \quad \text{... (1)}$$

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Omitting the third fraction from the equation (1) we have,

$$\frac{dx}{yz} = \frac{dy}{zx} \quad i.e., \frac{dx}{y} = \frac{dy}{x} \quad i.e., x\,dx = y\,dy \qquad ...(2)$$

On integrating (2), we have, $x^2 = y^2 + A$ ...(3)

A being an arbitrary constant.

Again omitting the first fraction from the equation (1) we have,

$$\frac{dy}{zx} = \frac{dz}{xy} \quad i.e., \frac{dy}{z} = \frac{dz}{y} \quad i.e., y\,dy = z\,dz \qquad ...(4)$$

On integrating (4) we have, $y^2 = z^2 + B$ ...(5)

B being an arbitrary constant.

Hence (3) and (5) together gives required general solution of the given differential equation (1).

15. Solve : $\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{z + \tan(3x + 2y)}$

Solution Here the given differential equation is

$$\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{z + \tan(3x + 2y)} \qquad ...(1)$$

Omitting the third fraction we have from (1),

$$\frac{dx}{2} = \frac{dy}{-3} \quad i.e., 3\,dx + 2\,dy = 0 \qquad ...(2)$$

Integrating (2) we have, $3x + 2y = A$

where A is an arbitrary constant. ...(3)

Again omitting the second fraction of differential equation (1) we have,

$$\frac{dx}{2} = \frac{dz}{z + \tan(3x + 2y)} \quad i.e., \frac{dx}{2} = \frac{dz}{z + \tan A} \quad \text{[Using (3)]} \qquad ...(4)$$

Integrating (4) we get,

$\frac{x}{2} = \log(z + A) + \frac{B}{2}$, where B is an arbitrary constant

i.e., $x = 2\log(3x + 2y + z) + B$ [From (3)] ...(5)

Hence from (3) and (5), the required general solutions are :

$3x + 2y = A$ and $x = 2\log(3x + 2y + z) + B$,

where A and B are two arbitrary constants.

16. Solve : $\frac{dx}{z(x + y)} = \frac{dy}{z(x - y)} = \frac{dz}{x^2 + y^2}$ [KH 2000]

Solution Here the given differential equation is

$$\frac{dx}{z(x + y)} = \frac{dy}{z(x - y)} = \frac{dz}{x^2 + y^2} \qquad ...(1)$$

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Multiplying the numerators and denominators of the three fractions of (1) by $x, -y, -z$ respectively we have,

$$\frac{x\,dx}{xz(x+y)} = \frac{-y\,dy}{-yz(x-y)} = \frac{-z\,dz}{-z(x^2+y^2)} = \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

i.e., $x\,dx - y\,dy - z\,dz = 0$

Integrating we get, $x^2 - y^2 - z^2 = A$ ...(2)

where A is an arbitrary constant.

Next multiplying the numerators and denominators of the three fractions of (1) by y, x and $-z$ respectively we have,

$$\frac{y\,dx}{yz(x+y)} = \frac{x\,dy}{xz(x-y)} = \frac{-z\,dz}{-z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}$$

i.e., $y\,dx + x\,dy - z\,dz = 0$ i.e., $d(xy) - z\,dz = 0$

Integrating we get, $xy - \frac{z^2}{2} = \frac{B}{2}$

i.e., $2xy - z^2 = B$, where B is an arbitrary constant. ...(3)

Combining (2) and (3), the general solution of the given differential equation can be written as

$x^2 - y^2 - z^2 = A$ and $2xy - z^2 = B$, where A and B are two arbitrary constant.

17. Solve : $\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}$ [KH 2003]

Solution Here the given differential equation is

$$\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z} \quad \text{... (1)}$$

Multiplying the numerators and denominators of 2nd and 3rd fraction of (1) respectively by y and z, we have,

$$\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{y\,dy}{y(y+z)} = \frac{z\,dz}{z(y-z)} = \frac{x\,dx + y\,dy + z\,dz}{z^2 - 2yz - y^2 + y(y+z) + z(y-z)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e., $x\,dx + y\,dy + z\,dz = 0$

Integrating we have, $x^2 + y^2 + z^2 = A$, where A is an arbitrary constant. ...(2)

Again from 2nd and 3rd fraction of (1) we have,

$\frac{dy}{y+z} = \frac{dz}{y-z}$ i.e., $(y+z)dz - (y-z)dy = 0$

i.e., $(y\,dz + z\,dy) + z\,dz - y\,dy = 0$ i.e., $d(yz) - y\,dy + z\,dz = 0$

Integrating we have, $2yz - y^2 + z^2 = B$, where B is an arbitrary constant. ...(3)

Hence from (2) and (3), the general solution of the given differential equation (1) is

$x^2 + y^2 + z^2 = A$ and $2yz - y^2 + z^2 = B$, where A and B are two arbitrary constants.

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18. Solve : $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$ [BH 2012; JH 19..]

Solution Here the given differential equation is

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz} \quad ...(1)$$

From the second and third fraction of (1) we have,

$$\frac{dy}{2xy}=\frac{dz}{2xz} \quad i.e., \quad \frac{dy}{y}=\frac{dz}{z}$$

Integrating we get, $\log y = \log Az$, A being an arbitrary constant.

i.e., $y = Az$ ...(2)

Again multiplying the numerators and denominators of each fractions of (1) by x, y and z respectively we have,

$$\frac{x\,dx}{x(x^2-y^2-z^2)}=\frac{y\,dy}{2xy^2}=\frac{z\,dz}{2xz^2}=\frac{x\,dx+y\,dy+z\,dz}{x(x^2-y^2-z^2)+2xy^2+2xz^2}$$

i.e., $\frac{y\,dy}{2xy^2}=\frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$ [considering 2nd and 4th fraction]

i.e., $\frac{dy}{2y}=\frac{d(x^2+y^2+z^2)}{2(x^2+y^2+z^2)}$ *i.e.*, $\frac{dy}{y}=\frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$

Integrating we get,

$\log(x^2+y^2+z^2) = \log By$, where B is an arbitrary constant.

i.e., $x^2+y^2+z^2 = By$ ...(3)

From (2) and (3), the required general solution of the differential equation (1) can be written as :

$y = Az$ and $x^2+y^2+z^2 = By$, where A and B are two arbitrary constants.

19. Solve : $\frac{dx}{x^2+y^2}=\frac{dy}{2xy}=\frac{dz}{(x+y)z}$ [BH 2013]

Solution Here the given differential equation is

$$\frac{dx}{x^2+y^2}=\frac{dy}{2xy}=\frac{dz}{(x+y)z} \quad ...(1)$$

Considering first two fractions of the equation (1), we have,

$$\frac{dx}{x^2+y^2}=\frac{dy}{2xy} \quad i.e., \quad (x^2+y^2)dy = 2xy\,dx$$

i.e., $y^2dy + (x^2dy - 2xy\,dx) = 0$ *i.e.*, $dy = \frac{2xy\,dx - x^2dy}{y^2}$ *i.e.*, $dy = d\left(\frac{x^2}{y}\right)$

Integrating we get, $y + A = \frac{x^2}{y}$, A being an arbitrary constant.

i.e., $x^2 = y^2 + Ay$ *i.e.*, $x^2 - y^2 = Ay$ ...(2)

Again multiplying the numerators and denomi[nators] of each [fractio]n of (1) [respec]tively by z, z and $(x+y)$ we have,

$$\frac{z\,dx}{z(x^2+y^2)} = \frac{z\,dy}{z\cdot 2xy} = \frac{(x+y)dz}{z(x+y)^2} = \frac{z\,dx + z\,dy - (x+y)dz}{z(x^2+y^2)+z\cdot 2xy-(x+y)^2\cdot z} = \frac{z\,dx+z\,dy-(x+y)dz}{0}$$

i.e., $z(dx + dy) - (x + y)dz = 0$ i.e., $\frac{d(x+y)}{x+y} = \frac{dz}{z}$

Integrating we get,

$\log(x + y) = \log(Bz)$, where B is an arbitrary constant.

i.e., $x + y = Bz$ ...(3)

Combining (2) and (3), the general solution of the given differential equation is

$x^2 - y^2 = Ay$ and $x + y = Bz$, where A and B are two arbitrary constants.

20. Solve : $\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z}$

Solution Here the given differential equation is $\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z}$ ...(1)

From the 1st and 2nd fraction of (1) we have,

$\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)}$ i.e., $\frac{dy}{dx} = \cot(x + y)$ ...(2)

To solve (2) we substitute, $v = x + y$.

Then we have, $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ giving $\frac{dy}{dx} = \frac{dv}{dx} - 1$

From (2) we have,

$\frac{dv}{dx} - 1 = \cot v$ i.e., $\frac{dv}{1+\cot v} = dx$ i.e., $dx = \frac{\sin v\,dv}{\sin v + \cos v} = \frac{1}{2}\frac{(\sin v - \cos v)+(\sin v + \cos v)}{\sin v + \cos v}dv$

i.e., $dx = \frac{1}{2}\left[dv - \frac{\cos v - \sin v}{\sin v + \cos v}dv\right]$ i.e., $2dx = dv - \frac{d(\sin v + \cos v)}{\sin v + \cos v}$

Integrating we have, $2x = v - \log(\sin v + \cos v) - \log A$, where A is an arbitrary constant.

i.e., $\log A(\sin v + \cos v) = v - 2x$

i.e., $A[\sin(x + y) + \cos(x + y)] = e^{y-x}$ [since $v = x + y$] ...(3)

Again from (1) we have,

$\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z} = \frac{dx+dy}{\sin(x+y)+\cos(x+y)}$ ...(4)

Considering 3rd and 4th fraction of (4), we have,

$\frac{dz}{z} = \frac{d(x+y)}{\sin(x+y)+\cos(x+y)} = \frac{dv}{\sin v + \cos v}$ [since $v = x + y$]

$= \frac{dv}{\sqrt{2}(\sin v \cos\frac{\pi}{4} + \cos v \sin\frac{\pi}{4})} = \frac{dv}{\sqrt{2}\sin(v+\frac{\pi}{4})}$ i.e., $\frac{\sqrt{2}\,dz}{z} = \operatorname{cosec}(v+\frac{\pi}{4})dv$

Integrating we get,

$\sqrt{2}\log z = \log B - \log\{\operatorname{cosec}(v+\frac{\pi}{4}) + \cot(v+\frac{\pi}{4})\}$, where B is arbitrary constant.

i.e., $z^{\sqrt{2}}\{\operatorname{cosec}(x+y+\frac{\pi}{4})\cot(x+y+\frac{\pi}{4})\} = B$ [since $v = x + y$] ...(5)

Hence the required general solution of the given differential equation can be obtained by combining the equations (3) and (5).

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$$\frac{z\,dx}{z(x^2+y^2)} = \frac{z\,dy}{z\cdot 2xy} = \frac{(x+y)dz}{z(x+y)^2} = \frac{z\,dx + z\,dy - (x+y)dz}{z(x^2+y^2) + z\cdot 2xy - (x+y)^2\cdot z} = \frac{z\,dx + z\,dy - (x+y)dz}{0}$$

i.e., $z(dx + dy) - (x + y)dz = 0$ i.e., $\frac{d(x+y)}{x+y} = \frac{dz}{z}$

Integrating we get,

$\log(x + y) = \log(Bz)$, where B is an arbitrary constant.

i.e., $x + y = Bz$ ...(3)

Combining (2) and (3), the general solution of the given differential equation is

$x^2 - y^2 = Ay$ and $x + y = Bz$, where A and B are two arbitrary constants.

20. Solve : $\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z}$

Solution Here the given differential equation is $\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z}$ ...(1)

From the 1st and 2nd fraction of (1) we have,

$\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)}$ i.e., $\frac{dy}{dx} = \cot(x + y)$ ...(2)

To solve (2) we substitute, $v = x + y$.

Then we have, $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ giving $\frac{dy}{dx} = \frac{dv}{dx} - 1$

From (2) we have,

$\frac{dv}{dx} - 1 = \cot v$ i.e., $\frac{dv}{1+\cot v} = dx$ i.e., $dx = \frac{\sin v\,dv}{\sin v + \cos v} = \frac{1}{2}\frac{(\sin v - \cos v) + (\sin v + \cos v)}{\sin v + \cos v}dv$

i.e., $dx = \frac{1}{2}\left[dv - \frac{\cos v - \sin v}{\sin v + \cos v}dv\right]$ i.e., $2dx = dv - \frac{d(\sin v + \cos v)}{\sin v + \cos v}$

Integrating we have, $2x = v - \log(\sin v + \cos v) - \log A$, where A is an arbitrary constant.

i.e., $\log A(\sin v + \cos v) = v - 2x$

i.e., $A\{\sin(x + y) + \cos(x + y)\} = e^{y-x}$ [since $v = x + y$] ...(3)

Again from (1) we have,

$\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z} = \frac{dx+dy}{\sin(x+y)+\cos(x+y)}$ ...(4)

Considering 3rd and 4th fraction of (4), we have,

$\frac{dz}{z} = \frac{d(x+y)}{\sin(x+y)+\cos(x+y)} = \frac{dv}{\sin v + \cos v}$ [since $v = x + y$]

$= \frac{dv}{\sqrt{2}(\sin v\cos\frac{\pi}{4} + \cos v\sin\frac{\pi}{4})} = \frac{dv}{\sqrt{2}\sin(v+\frac{\pi}{4})}$ i.e., $\frac{\sqrt{2}\,dz}{z} = \operatorname{cosec}(v+\frac{\pi}{4})dv$

Integrating we get,

$\sqrt{2}\log z = \log B - \log\{\operatorname{cosec}(v+\frac{\pi}{4}) + \cot(v+\frac{\pi}{4})\}$, where B is arbitrary constant.

i.e., $z^{\sqrt{2}}\{\operatorname{cosec}(x+y+\frac{\pi}{4})\cot(x+y+\frac{\pi}{4})\} = B$ [since $v = x + y$] ...(5)

Hence the required general solution of the given differential equation can be obtained by combining the equations (3) and (5).

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21. Solve : $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$

Solution Here the given differential equation is

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2} \qquad ...(1)$$

Considering first two fractions of differential equation (1), we have,

$$\frac{dx}{xy} = \frac{dy}{y^2} \quad i.e., \quad \frac{dx}{x} = \frac{dy}{y}$$

Integrating we get, $\log x = \log A + \log y$, A being an arbitrary constant.

i.e., $x = Ay$ ...(2)

Again considering 1st and 3rd fractions of the differential equation (1), we have,

$$\frac{dx}{xy} = \frac{dz}{xyz - 2x^2} \quad i.e., \quad \frac{dx}{1} = \frac{dz}{z - 2\frac{x}{y}} = \frac{dz}{z - 2A} \quad \text{giving } dx = \frac{dz}{z - 2A} \quad \text{[From (2)]}$$

Integrating we get, $x = \log(z - 2A) + B$, where B is an arbitrary constant.

i.e., $x = \log\left(z - \frac{2x}{y}\right) + B$ [From (2)] ...(3)

Combining (2) and (3), the general solution of the given differential equation can be written as : $x = Ay$ and $x = \log\left(z - \frac{2x}{y}\right) + B$; where A and B are two arbitrary constants.

Exercise

1. From the system of equation $\frac{dx}{dt} + 5x + y = e^t$, $\frac{dy}{dt} - x + 3y = e^{2t}$ obtain the differential equation of the two variables x and t.
2. From the system of equations $\frac{dx}{dt} + 2y = 0$, $\frac{dy}{dt} - 2y = x$ obtain the differential eqation of the two variables y and t.
3. Given $\frac{dx}{dt} - 3x - 2y = 0$, $\frac{dy}{dt} - 5x - 3y = 0$. Express x in terms of t.
4. Given $\frac{dx}{dt} = 5x + 4y$, $\frac{dy}{dt} = -x + y$. Express y in terms of t.
5. Solve : $\frac{dx}{dt} + 2y = 0$, $\frac{dy}{dt} - x = 0$.
6. Solve : $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0$, $\frac{dy}{dt} + 5x + 3y = 0$. **[CH 1992, 1994]**
7. Solve : $\frac{dx}{dt} + 2y = 0$, $\frac{dy}{dt} - 2y = x$.
8. Solve : $\frac{dx}{dt} + 3x + y = e^t$, $\frac{dy}{dt} - x + y = e^{2t}$.
9. Solve : $\frac{dx}{dt} + 2x + 3y = 0$, $\frac{dy}{dt} + 3x + 2y = 2e^{3t}$.
10. Solve : $\frac{dx}{dt} + y = e^t$, $\frac{dy}{dt} - x = e^{-t}$.
11. Solve : $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$.
12. Solve : $\frac{dx}{dt} + \frac{dy}{dt} + 5x + 3y = e^{-t}$, $2\frac{dx}{dt} + \frac{dy}{dt} + x + y = 3$
13. Solve: $\frac{dx}{dt} + x - 3y = 0$, $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \sin t$

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14. Solve : $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$

$\frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t$

with $x(0) = 0$, $y(0) = 1$.

15. Solve : $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

16. Solve : $\frac{dx}{\sin^2 x} = \frac{dy}{\sin^2 y} = \frac{dz}{\sin^2 z}$

17. Solve : $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$

[KU (Hons) 2000]

18. Solve : $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

19. Solve : $\frac{a\,dx}{bc(y-z)} = \frac{b\,dy}{ca(z-x)} = \frac{c\,dz}{ab(x-y)}$

20. Solve : $\frac{l\,dx}{(x-n)yz} = \frac{m\,dy}{(n-l)zx} = \frac{n\,dz}{(l-m)xz}$

Answers

1. $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 2e^t - e^{2t}$

2. $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$

3. $x = A\cos t + B\sin t$

4. $y = (A + Bt)e^{3t}$

5. $x = -A\sqrt{2}\sin\sqrt{2}t + B\sqrt{2}\cos\sqrt{2}t$

$y = -A\cos\sqrt{2}t + B\sin\sqrt{2}t$

6. $x = A\cos t + B\sin t$,

$y = -\frac{1}{2}(A + 3B)\sin t + \frac{1}{2}(B - 3A)\cos t$

7. $x = -Ae^t + (1 - t)e^t$

$y = (A + Bt)e^t$

8. $x = (A + Bt)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$

$y = -\{A + (1 + t)B\}e^{-2t} + \frac{1}{9}e^t + \frac{5}{16}e^{2t}$

9. $x = Ae^t + Be^{-5t} - \frac{3}{8}e^{3t}$

$y = -Ae^t - Be^{-5t} + \frac{5}{8}e^{3t}$

10. $x = A\cos t + B\sin t + \frac{1}{2}(e^t - e^{-t})$

$y = A\sin t - B\cos t + \frac{1}{2}(e^t - e^{-t})$

11. $x = Ae^t + Be^{-t}$, $y = \sin t - Ae^t + Be^{-t}$

12. $x = -\frac{2}{3}Ae^t - \frac{B}{3}e^{-2t} - \frac{9}{2}$

$y = Ae^t + Be^{-2t} + \frac{15}{2} + \frac{e^{-t}}{2}$

13. $x = \frac{3}{2}Ae^{2t} - 3Be^{-t} - \frac{3}{10}(\cos t - 2\sin t)e^t$

$y = Ae^t + Be^{-2t} - \frac{1}{10}(\cos t + 3\sin t)$

14. $x = e^t\left(\frac{1}{2}\cos t - \sin t\right) - \frac{1}{2}\cos 2t$

$y = e^t\left(\frac{1}{2}\sin t + \cos t\right) - \frac{1}{2}\sin 2t$

15. $\frac{x}{y} = A$, $\frac{y}{z} = B$, where A and B are two arbitrary constants.

16. $a + x = a + y + A$, $a + x = a + z + B$

17. $x^2 - y^2 - z^2 = A$, $2xy - z^2 = B$

18. $x + y + z = A$, $xyz = B$

19. $a^2x + b^2y + c^2z = A$,

$a^2x^2 + b^2y^2 + c^2z^2 = B$

20. $lx^2 + my^2 + nz^2 = A$,

$l^2x^2 + m^2y^2 + n^2z^2 = B$

[where A and B are two arbitrary constants in each case.]

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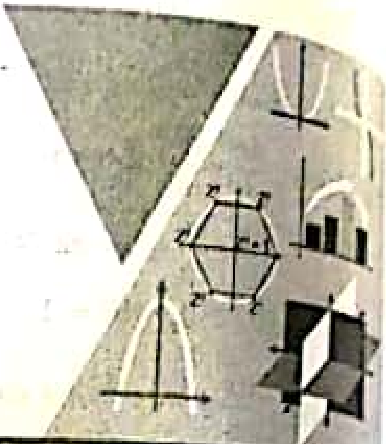
12 CHAPTER

Total Differential Equations



Generally a differential equation of the form $p(x, y, z)dx + q(x, y, z)dy + r(x, y, z)dz = 0$ ... (1) is called a total differential equation of three variables x, y, z. The equation (1) is also known as the Pfaffian differential equation. Like other first order differential equations, the equation (1) can also be solved if and only if it satisfy the condition of integrability. In particular, the **condition for integrability** for the differential equation (1) is

$$p\left(\frac{\partial q}{\partial z}-\frac{\partial r}{\partial y}\right)+q\left(\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}\right)+r\left(\frac{\partial p}{\partial x}-\frac{\partial q}{\partial y}\right)=0 \quad \text{... (2)}$$

It should be noted that the condition (2) is both necessary as well as sufficient for having a solution of the differential equation (1). In other words, like an exact differential equation of two variables, here also the solution of the differential equation (1) entirely depends on the condition (2). The solution of the differential equation (1) should be a function of all three varibles x, y, z, say, $f(x, y, z) = c$ ... (3)

where c is an arbitrary constant. Further we can easily show that (3) is a solution of the equation (1) since taking differentiation of (3), we have $d\,f(x, y, z) = 0$

$$i.e., \quad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \quad \text{... (4)}$$

where the equation (1) is the same form of (4).

Solution Procedure : Here our target is to solve the differential equation (1). When the condition for integrability (2) is satisfied then we can solve (1). There are various method to solve the differential equation (1) and those methods are described as below.

- **Method-1 (By Inspection) :** In most problems on total differential equation, we can get solution easily by rearranging the differential coefficients. In that situation we can easily prepare the differential equation (1) into a total differential whose antiderivative would provide the general solution containing one arbitrary constant.

Illustration-1 : Solve the differential equation $(yz + 4x^3)dx + (zx + 3y^2)dy + (xy + 2z)dz = 0$

Compairing the given differential equation

$(yz + 4x^3)dx + (zx + 3y^2)dy + (xy + 2z)dz = 0$ ... (5)

with its standard form $pdx + qdy + rdz = 0$, we have, $p = yz + 4x^3$, $q = zx + 3y^2$ and $r = xy + 2z$.

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Now we have, $p\left(\frac{\partial q}{\partial z}-\frac{\partial r}{\partial y}\right)+q\left(\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}\right)+r\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)$

$= (yz + 4x^3)(x - x) + (zx + 3y^2)(y - y) + (xy + 2z)(z - z) = 0$

So, we see that the condition for integrability has been satisfied for the present problem (5). Now by inspection, the equation (5) can be rearranged as follows :

$(yz\,dx + zx\,dy + xy\,dz) + (4x^3dx + 3y^2dy + 2zdz) = 0$

i.e., $d(xyz) + d(x^4) + d(y^3) + d(z^2) = 0$

i.e., $d(xyz + x^4 + y^3 + z^2) = 0$

Integrating we have, $xyz + x^4 + y^3 + z^2 = c$, c being an arbitrary constant.

Hence the complete solution of the given differential equation is $xyz + x^4 + y^3 + z^2 = c$, where c is an arbitrary constant.

- **Method-2 : Solution technique for homogeneous differential equation.**

We recall the differential equation (1) again as below

$pdx + qdy + rdz = 0$

Now suppose all three functions of p, q and r are homogeneous functions of x, y and z of same order. Then we can solve the above mentioned homogeneous differential equation by substituting $x = uz$ and $y = vz$ so that we have $dx = u\,dz + z\,du$ and $dy = v\,dz + z\,dv$. Using these in the homogeneous differential equation, the given differential equation reduces to $p_1(u, v)du + p_2(u, v)dv = 0$ ... (6)

Equation (6) is a first order differential equation of two variables which can be solved easily. If the general solution of (6) becomes as $f_1(u, v) = c$, where c is an arbitrary constant then the complete solution of the given homogeneous differential equation is $f_1(x/z, y/z) = c$.

Illustration-2 : Solve $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$.

Comparing the given differential equation to its standard form $pdx + qdy + rdz = 0$ we have

$p = 2xz - yz,\ q = 2yz - zx;\ r = -(x^2 - xy + y^2)$

We have, $p\left(\frac{\partial q}{\partial z}-\frac{\partial r}{\partial y}\right)+q\left(\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}\right)+r\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)$

$= (2xz - yz)(2y - x + x - y) + (2yz - zx)(-2x + y - 2x + y) - (x^2 - xy + y^2)(-z + z) = 0.$

Since the condition of integrability has been satisfied and all of p, q and r are homogeneous function of order 2, then we can solve by substituting $x = uz, y = vz$ so that $dx = u\,dz + z\,du$ and $dy = v\,dz + z\,dv$ and the given differential equation reduces as

$(2uz^2 - vz^2)(udz + zdu) + (2vz^2 - uz^2)(vdz + zdv) - (u^2z^2 - uvz^2 + v^2z^2)dz = 0$

i.e., $z(2u - v)du + z(2v - u)dv + (u^2 - uv + v^2)dz = 0$

i.e., $z[2udu + 2vdv - (u\,dv + v\,du)] + (u^2 - uv + v^2)dz = 0$

i.e., $zd(u^2 - uv + v^2) + (u^2 - uv + v^2)dz = 0$

i.e., $d[z(u^2 - uv + v^2)] = 0$

Integrating we have, $z(u^2 - uv + v^2) = c$, where c is an arbitrary constant.

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Now substituting $u = \frac{x}{z}$ and $v = \frac{y}{z}$, the complete solution for original problem becomes as : $z\left(\frac{x^2}{z^2} - \frac{xy}{z^2} + \frac{y^2}{z^2}\right) = c$

i.e., $x^2 - xy + y^2 = cz$.

- **Method-3 : Choosing one variable as constant :**

We recall the differential equation (1) as follows :

$$pdx + qdy + rdz = 0$$

Among x, y and z, let us choose one of them as constant, say z = constant. Then $dz = 0$ and the above equation reduces to $pdx + qdy = 0$, which can be solved easily by considering z, appearing in p and q as constant. Let the solution $pdx + qdy = 0$ becomes $f_2(x, y)$ = constant and this constant should be independent of both x and y but may be dependent on third variable z so that we can consider $f_2(x, y) = f_3(z)$, where $f_3(z)$ is an unknown function. To determine $f_3(z)$, we have to differentiate the relation $f_2(x, y) = f_3(z)$ and then require to compare with original equation to get the required solution.

Illustration-3 : Solve : $ydx - 2x\,dy - y^3z\,dz = 0$

Here the given differential equation is :

$$ydx - 2x\,dy - y^3z\,dz = 0 \quad \text{... (7)}$$

To solve (7), we choose z = constant so that $dz = 0$ and equation (7) reduces as :

$$ydx - 2x\,dy = 0$$

i.e., $-\frac{dx}{x} + \frac{2dy}{y} = 0$

Integrating we have, $-\log x + 2\log y = \log A$, where A is a constant and dependent of x and y both but may be dependent on z, so that we may consider $A = \phi(z)$. Then we have from above,

$$\frac{y^2}{x} = A = \phi(z) \quad \text{... (8)}$$

To determine the unknown function ϕ, taking differential of (8), we have,

$$\frac{x \cdot 2ydy - y^2dx}{x^2} - \phi'(z)dz = 0$$

i.e., $-\frac{y^2}{x^2}dx + \frac{2y}{x}dy - \phi'(z)dz = 0$

i.e., $y^2dx - 2xy\,dy + x^2\phi'(z)dz = 0 \quad \text{... (9)}$

Clearly (7) and (9) represents same equation and so comparing (7) and (9) we have,

$\frac{y}{y^2} = \frac{-2x}{-2xy} = \frac{-y^3z}{x^2\phi'(z)}$ i.e., $\phi'(z) = -\frac{y^4z}{x^2} = -z\left(\frac{y^2}{x}\right)^2 = -z[\phi(z)]^2$ [From 8]

i.e., $\frac{d}{dz}\phi(z) + z[\phi(z)]^2 = 0$ i.e., $\frac{d\phi(z)}{[\phi(z)]^2} + zdz = 0$

Integrating we have, $-\frac{1}{\phi(z)} + \frac{z^2}{2} + c = 0$, where c is an arbitrary constant,

i.e., $\frac{1}{\phi(z)} = \frac{z^2}{2} + c,$

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From (8), the required complete solution becomes as :

$\frac{x}{y^2} = \frac{z^2}{2} + c$, where c is an arbitrary constant.

Note: This type of problems can be solved even considering either x = constant or y = constant, according to the necessity of the problem.

- **Method-4 : Solution with the help of auxiliary equations.**

We recall the differential equation (1) as below

$$p\,dx + q\,dy + r\,dz = 0$$

and we solve this differential equation when we cannot apply any of the previous methods to solve it. Now the condition of integrability gives,

$$p\left(\frac{\partial q}{\partial z}-\frac{\partial r}{\partial y}\right)+q\left(\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}\right)+r\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)=0$$

Now comparing above two equations we have, ... (9)

$$\frac{dx}{\frac{\partial q}{\partial z}-\frac{\partial r}{\partial y}}=\frac{dy}{\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}}=\frac{dz}{\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}}$$

Equation (9) is known as the auxiliary equations. From (9) we will get two integrals $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. Then our target is to obtain two functions $\phi(u, v)$ and $\psi(u, v)$ such that the equation (1) can be re-written as ... (10)

$$\phi(u, v)du + \psi(u, v)dv = 0$$

where $\phi(u, v)$ and $\psi(u, v)$ can be determined by comparing (1) and (10). After that substituting these values in (10) and integrating we have required general solution containing one arbitrary constant. Below we present some well known differentials which are to be used subsequently in this chapter to solve various problems.

Some Important Differentials

1. $x\,dy + y\,dx = d(xy)$
2. $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$
3. $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$
4. $\frac{xdy - ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$
5. $\frac{ydx - xdy}{x^2+y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
6. $\frac{xdx + ydy}{x^2+y^2} = \frac{1}{2}d[\log(x^2+y^2)]$
7. $\frac{xdy - ydx}{xy} = d\left(\log\frac{y}{x}\right)$
8. $\frac{xdy + ydx}{xy} = d[\log(xy)]$
9. $-\frac{xdy + ydx}{(xy)^2} = d\left(\frac{1}{xy}\right)$
10. $\frac{2x^2ydy - 2xy^2dx}{x^4} = d\left(\frac{y^2}{x^2}\right)$
11. $\frac{ye^xdx - e^xdy}{y^2} = d\left(\frac{e^x}{y}\right)$
12. $2xdx + 2ydy = d(x^2+y^2)$
13. $2xdx + 2ydy + 2zdz = d(x^2+y^2+z^2)$
14. $xydz + yzdx + zxdy = d(xyz)$

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Illustrative Examples

1. **Show that $(2x + y^2 + 2zx)dx + 2xydy + x^2dz = 0$ is integrable.**

[Bangalore 1995]

Solution Here the given differential equation is

$$(2x + y^2 + 2zx)dx + 2xydy + x^2dz = 0 \quad \dots (1)$$

Comparing (1) with the standard form $pdx + qdy + rdz = 0$, we have

$p = (2x + y^2 + 2zx)$, $q = 2xy$, $r = x^2$.

$$\text{Now, } p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$$

$$= (2x + y^2 + 2xz)(0 - 0) + 2xy(2x - 2x) + x^2(2y - 2y) = 0$$

Hence the condition for integrability is satisfied.

2. **Test for the integrability condition of the differential equation $xdy - ydx - 2x^2zdz = 0$**

Solution Here the given differential equation is $-ydx + xdy - 2x^2z = 0 \quad \dots (1)$

Comparing (1) with the standard form $pdx + qdy + r\,dz = 0$,

we have $p = -y$, $q = x$, $r = -2x^2z$

$$\text{Now, } p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$$

$$= -y(0 - 0) + x(-4xz - 0) - 2x^2z(-1 - 1) = -4x^2z + 4x^2z = 0.$$

Thus the condition for integrability is satisfied.

3. **Solve $x(y + z)dx + dy + dz = 0$**

Solution Here the given differential equation is $x(y + z)dx + dy + dz = 0 \quad \dots (1)$

Comparing (1) with its standard form $pdx + qdy + rdz = 0$, we have, $p = x(y + z)$, $q = 1$, $r = 1$.

$$\text{Now we have, } p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = x(y + z)(0 - 0) + 1(0 - x) + 1(x - 0) = 0$$

Thus the condition for integrability is satisfied and by inspection we may solve the equation (1).

Thus from (1) we have, $xdx + \dfrac{dy + dz}{y + z} = 0$ *i.e.,* $x\,dx + \dfrac{d(y + z)}{y + z} = 0$

Integrating we have, $\dfrac{x^2}{2} + \log(y + z) = \dfrac{c}{2}$, where c is an arbitrary constant

i.e., $c - x^2 = 2\log(y + z)$

which is the required general solution.

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4. **Solve : $xydz + yzdx + zxdy = 0$**

Solution Here the given differential equation is :

$xydz + yzdx + zxdy = 0$ ... (1)

Comparing (1) with its standard form $pdx + qdy + rdz = 0$, we have,

$p = yz,\ q = zx,\ r = xy$

Now we have, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = yz(x - x) + zx(y - y) + xy(z - z) = 0$

Thus the condition for integrability is satisfied and by inspection, we can solve the equation (1)

We have $d(xyz) = xydz + yzdx + zxdy = 0$.

Thus the equation (1) reduces to :

$d(xyz) = 0$, which on integration becames

$xyz = c$, where c is an arbitrary constant and this is the required general solution for the given problem.

5. **Solve : $ydx + xdy + xyzdz = 0$**

Solution Here, the given differential equation is $ydx + xdy + xyzdz = 0$ ... (1)

Comparing (1) with its standard form $pdx + qdy + rdz = 0$,

we have, $p = y,\ q = x,\ r = xyz.$

Now we have,

$$p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = y(0 - xz) + x(yz - 0) + xyz(1 - 1) = 0.$$

Thus the condition for integrability is satisfied and the differential equation (1) can be solved by inspection.

We have from (1), $ydx + xdy + xyz\,dz = 0$

$d(xy) + xyz\,dz = 0$ *i.e.*, $\frac{d(xy)}{xy} + zdz = 0$

Integrating we have, $\log(xy) + \frac{z^2}{2} = \log c$, c being an arbitrary constant

i.e., $z^2 = 2\log\left(\frac{c}{xy}\right)$, which is the required general solution.

6. **Solve : $(y^2 + z^2)dx + 2xydy + 2xzdz = 0.$**

Solution Here the given differential equation is $(y^2 + z^2)dx + 2xydy + 2xzdz = 0$ ... (1)

Comparing (1) with the standard form $pdx + qdy + rdz = 0$, we have $p = y^2 + z^2$, $q = 2xy$ and $r = 2xz$.

$$\text{Now, } p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$$
$$= (y^2 + z^2)(0 - 0) + 2xy(2z - 2z) + 2xz(2y - 2y) = 0$$

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Thus the condition of integrability is satisfied and the differential equation (1) can be solved by inspection.

From (1) we have,

$(y^2 + z^2)dx + x(2ydy + 2zdz) = 0$ *i.e.*, $(y^2 + z^2)dx + xd(y^2 + z^2) = 0$

i.e., $d[x(y^2 + z^2)] = 0$

Integrating we have, $x(y^2 + z^2) = c$, where c is an arbitrary constant.

Hence the required general solution for the given differential equation is $x(y^2 + z^2) = c$.

7. Solve : $(y^2 + z^2)dx + z(1 + x^2)dy - y(1 + x^2)dz = 0$

Solution Here the given differential equation is :

$$(y^2 + z^2)dx + z(1 + x^2)dy - y(1 + x^2)dz = 0 \quad \text{... (1)}$$

Comparing the differential equation (1) with its standard form $pdx + qdy + rdz = 0$, we have, $p = y^2 + z^2$, $q = z(1 + x^2)$, $r = -y(1 + x^2)$

Now, we ahve, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= (y^2 + z^2)[(1 + x^2) + (1 + x^2)] + z(1 + x^2)(-2xy - 2z) - y(1 + x^2)(2y - 2zx)$

$= (1 + x^2)(2y^2 + 2z^2 - 2z^2 - 2xyz - 2y^2 + 2xyz) = 0.$

Thus the condition for integrability is satisfied and the given equation (1) can be solved by inspection. So from (1) we have,

$(y^2 + z^2)dx + (1 + x^2)(zdy - ydz) = 0$

i.e., $\frac{dx}{1+x^2} + \frac{z^2 d\left(\frac{y}{z}\right)}{y^2 + z^2} = 0$ *i.e.*, $\frac{dx}{1+x^2} + \frac{d\left(\frac{y}{z}\right)}{1+\left(\frac{y}{z}\right)^2} = 0$

Integrating we have,

$\tan^{-1}x + \tan^{-1}\frac{y}{z} = \tan^{-1}c$, where c is an arbitrary constant.

i.e., $\tan^{-1}\frac{x + \frac{y}{z}}{1 - \frac{xy}{z}} = \tan^{-1}c$

i.e., $y + xz = c(z - xy)$, which is the required general solution for c being an arbitrary constant.

8. Solve : $4x^3zdx + 4y^3zdy + (x^4 + y^4 + e^z)dz = 0$

Solution Here the given differential equation is

$$4x^3zdx + 4y^3zdx + (x^4 + y^4 + e^z)dz = 0 \quad \text{... (1)}$$

Comparing (1) with its standard form $pdx + qdy + rdz = 0$

we have, $p = 4x^3z$, $q = 4y^3z$ and $r = x^4 + y^4 + e^z$

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= 4x^3(4y^3 - 4y^3) + 4y^3(4x^3 - 4x^3) + (x^4 + y^4 + e^z)(0 - 0)$

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Thus the condition for integrability is satisfied and the given equation (1) can be solved by inspection. So from (1) we have,

$z(4x^3dx + 4y^3dy) + (x^4 + y^4)dz + e^z dz = 0$

i.e., $z d(x^4 + y^4) + (x^4 + y^4)dz + e^z dz = 0$

i.e., $d(z(x^4 + y^4)) + e^z dz = 0$

Integrating we have, $z(x^4 + y^4) + e^z = c$, where c is an arbitrary constant and this is the required general solution.

9. **Solve $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$** **[Delhi Maths (Hons) 2001]**

Solution Here the given differential equation is

$$(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0 \quad \text{... (1)}$$

Comparing (1) with the standard form

$pdx + qdy + rdz = 0$, we have

$p = 2x^2 + 2xy + 2xz^2 + 1$, $q = 1$, $r = 2z$.

Now $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= (2x^2 + 2xy + 2xz^2 + 1)(0 - 0) + 1(0 - 4xz) + 2z(2x - 0)$

$= -4zx + 4zx = 0$

Thus the condition for integrability is satisfied.

Now rearranging the given differential equation (1), we get

$2x(x + y + z^2)dx + (dx + dy + 2zdz) = 0$

i.e., $2x(x + y + z^2)dx + d(x + y + z^2) = 0$

i.e., $2xdx + \dfrac{d(x + y + z^2)}{x + y + z^2} = 0$

Now integrating we get, $x^2 + \log(x + y + z^2) = \log c$ where c is an arbitrary constant.

$(x + y + z^2)e^{x^2} = c$, which is the required general solution.

10. **Solve $yz\,dx + zx \log x\,dy - xy \log x\,dz = 0$**

Solution Here the given differential equation is

$$yzdx + zx\log x dy - xy\log x dz = 0 \quad \text{... (1)}$$

Comparing (1) with its standard form $pdx + qdy + rdz = 0$ we have, $p = yz$, $q = zx \log x$ and $r = -xy \log x$.

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= yz(x\log x + x \log x) + zx\log x(-y\log x - y - y) - xy\log x(z - z\log x - z)$

$= 2xyz\log x - 2xyz\log x - xyz(\log x)^2 + xyz(\log x)^2 - xy\log x(z - z) = 0.$

Thus the condition for integrability has been satisfied and the differential equation (1) can be solved by inspection. We have from (1),

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$yz\,dx + \log x(zx\,dy - xy\,dz) = 0$

i.e., $\frac{dx}{x} + \log x\left(\frac{dy}{y} - \frac{dz}{z}\right) = 0$ i.e., $\frac{d(\log x)}{\log x} + \frac{dy}{y} = \frac{dz}{z}$

Integrating we have, $\log z = \log y + \log(\log x) + \log c$ where c is an arbitrary constant.

i.e., $z = cy \log x$, which is the required general solution.

11. **Solve : $(a - z)(ydx + xdy) + xydz = 0$** **[Delhi (G) 2002]**

Solution Here the given differential equation is :

$(a - z)(ydx + xdy) + xydz = 0$

i.e., $(a - z)ydx + (a - z)xdy + xydz = 0$ ...(1)

Comparing (1) with its standard form $pdx + qdy + rdz = 0$,

We have, $p = (a - z)y$, $q = (a - z)x$, $r = xy$

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= (a - z)y(-x - x) + (a - z)x(y + y) + xy(a - z - a + z) = 0$

Thus the condition for integrability has been satisfied. So we have from (1),

$(a - z)(ydx + xdy) + xydz = 0$

i.e., $(a - z)d(xy) + xydz = 0$ i.e., $\frac{d(xy)}{xy} - \frac{dz}{z - a} = 0$

Integrating we have,

$\log(xy) - \log(z - a) = \log c$, c being an arbitrary constant.

i.e., $xy = c(z - a)$, which is the required general solution.

12. **Find $f(y)$ such that the total differential equation $[(yz + z)/x]dx - zdy + f(y)dz = 0$ is integrable.**

Solution Here the given differential equation is $\frac{yz + z}{x}dx - zdy + f(y)dz = 0$

i.e., $z(y + 1)dx - zxdy + xf(y)dz = 0$... (1) [Multiplying throughout by x]

Now comparing (1) with the standard form $pdx + qdy + rdz = 0$,

we have $p = z(y + 1)$, $q = -z\,x$, $r = x\,f(y)$

If equation (1) is integrable then

$$p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0$$

i.e., $z(y + 1)[-x - xf'(y)] - xz[f(y) - (y + 1)] + x\,f(y)\,[z - (-z)] = 0$

i.e., $xz(1 + y)f'(y) = xz\,f(y) \Rightarrow (1 + y)f'(y) = f(y)$

i.e., $\frac{d[f(y)]}{f(y)} = \frac{dy}{y+1}$

Now integrating, we get

$\log[f(y)] = \log(y + 1) + \log c$ where c is an arbitrary constant.

$\Rightarrow f(y) = c(y + 1)$, which is the required functional value.

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13. Solve : $\dfrac{y+z-2x}{(y-x)(z-x)}dx + \dfrac{z+x-2y}{(z-y)(x-y)}dy + \dfrac{x+y-2z}{(x-z)(y-z)}dz = 0$ [Delhi (G) 2004]

Solution Here the given differential equation is :

$$\frac{y+z-2x}{(y-x)(z-x)}dx + \frac{z+x-2y}{(z-y)(x-y)}dy + \frac{x+y-2z}{(x-z)(y-z)}dz = 0 \qquad \text{... (1)}$$

Comparing (1) with its standard form $p\,dx + q\,dy + r\,dz = 0$

we have, $p = \dfrac{y+z-2x}{(y-x)(z-x)}$, $q = \dfrac{z+x-2y}{(z-y)(x-y)}$, $r = \dfrac{x+y-2z}{(x-z)(y-z)}$

It can easily be verified that, $p\left(\dfrac{\partial q}{\partial z} - \dfrac{\partial r}{\partial y}\right) + q\left(\dfrac{\partial r}{\partial x} - \dfrac{\partial p}{\partial z}\right) + r\left(\dfrac{\partial p}{\partial y} - \dfrac{\partial q}{\partial x}\right) = 0$

Thus condition of integrability has been satisfied and so the equation (1) can be solved.

So, we have from (1)

$$\frac{(y-x)+(z-x)}{(y-x)(z-x)}dx + \frac{(z-y)+(x-y)}{(z-y)(x-y)}dy + \frac{(x-z)+(y-z)}{(x-z)(y-z)}dz = 0$$

i.e., $\left(\dfrac{1}{z-x} + \dfrac{1}{y-x}\right)dx + \left(\dfrac{1}{x-y} + \dfrac{1}{z-y}\right)dy + \left(\dfrac{1}{y-z} + \dfrac{1}{x-z}\right)dz = 0$

i.e., $\dfrac{1}{z-x}(dx - dz) + \dfrac{1}{x-y}(dy - dx) + \dfrac{1}{y-z}(dz - dy) = 0$

i.e., $\dfrac{dx-dy}{x-y} + \dfrac{dy-dz}{y-z} + \dfrac{dz-dx}{z-x} = 0$ i.e., $\dfrac{d(x-y)}{x-y} + \dfrac{d(y-z)}{y-z} + \dfrac{d(z-x)}{z-x} = 0$

Now Integrating, we get

$\log(x-y) + \log(y-z) + \log(z-x) = \log c$, where c is an arbitrary constant.

$(x-y)(y-z)(z-x) = c$, which is the required general solution.

14. Solve : $(y^2 + z^2 - x^2)dx - 2xy\,dy - 2xz\,dz = 0$ [NBH 2005]

Solution Here the given differential equation is

$$(y^2 + z^2 - x^2)dx - 2xy\,dy - 2xz\,dz = 0 \qquad \text{... (1)}$$

Comparing (1) with its standard form $p\,dx + q\,dy + r\,dz = 0$,

we have $p = y^2 + z^2 - x^2$, $q = -2xy$ and $r = -2xz$.

Now $p\left(\dfrac{\partial q}{\partial z} - \dfrac{\partial r}{\partial y}\right) + q\left(\dfrac{\partial r}{\partial x} - \dfrac{\partial p}{\partial z}\right) + r\left(\dfrac{\partial p}{\partial y} - \dfrac{\partial q}{\partial x}\right)$

$= (y^2 + z^2 - x^2)(0 - 0) - 2xy(-2z - 2z) - 2xz(2y + 2y) = 8xyz - 8xyz = 0.$

Thus the condition for integrability for the system (1) has been satisfied. Now the differential equation (1) can be solved by method of inspection. So from (1) we have,

$(y^2 + z^2)dx - x^2dx - x(2y\,dy + 2z\,dz) = 0$

i.e., $(y^2 + z^2)dx - x^2dx - x\,d(y^2 + z^2) = 0$ i.e., $\dfrac{(y^2 + z^2)dx - x\,d(y^2 + z^2)}{x^2} - dx = 0$

i.e., $d\left(\dfrac{y^2 + z^2}{x}\right) + dx = 0$ i.e., $d\left(x + \dfrac{y^2 + z^2}{x}\right) = 0$

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Integrating we have, $x + \frac{y^2+z^2}{x} = c$, where c is an arbitrary constant.

i.e., $x^2 + y^2 + z^2 = cx$, which is the required general solution for the given differential equation, with c being an arbitrary constant.

15. Solve : $\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\frac{y}{x}dz = 0$ [BH 2003, 2008]

Solution Here the given differential equation is

$$\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\frac{y}{x}dz = 0 \quad \text{... (1)}$$

Comparing (1) with its standard form $pdx + qdy + rdz = 0$,

we have, $p = \frac{yz}{x^2+y^2}$, $q = -\frac{xz}{x^2+y^2}$, $r = -\tan^{-1}\frac{y}{x}$

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$$= \frac{yz}{x^2+y^2}\left(-\frac{x}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}\right) - \frac{xz}{x^2+y^2}\left(\frac{y}{x^2}\cdot\frac{1}{1+\frac{y^2}{x^2}} - \frac{y}{x^2+y^2}\right)$$

$$- \tan^{-1}\frac{y}{x}\left(\frac{\partial}{\partial y}\left(\frac{yz}{x^2+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{xz}{x^2+y^2}\right)\right)$$

$$= \frac{yz}{x^2+y^2}\left(-\frac{x}{x^2+y^2} + \frac{x}{x^2+y^2}\right) - \frac{xz}{x^2+y^2}\left(\frac{y}{x^2+y^2} - \frac{y}{x^2+y^2}\right)$$

$$- \tan^{-1}\frac{y}{x}\left(\frac{z}{x^2+y^2} - \frac{2y^2z}{(x^2+y^2)^2} + \frac{z}{x^2+y^2} - \frac{2x^2z}{(x^2+y^2)^2}\right)$$

$$= -\tan^{-1}\frac{y}{x}\left(\frac{2z}{x^2+y^2} - \frac{2z(x^2+y^2)}{(x^2+y^2)^2}\right) = 0$$

Thus the condition of integrability is satisfied and we can solve the given equation by inspection. We have from (1),

$$\frac{z}{x^2+y^2}(y\,dx - x\,dy) - \tan^{-1}\frac{y}{x}dz = 0 \quad i.e., \; -\frac{zx^2 d\left(\frac{y}{x}\right)}{x^2+y^2} - \tan^{-1}\frac{y}{x}dz = 0$$

$$i.e., \; \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} + \tan^{-1}\frac{y}{x}\frac{dz}{z} = 0 \quad i.e., \; \frac{d\left(\tan^{-1}\frac{y}{x}\right)}{\tan^{-1}\frac{y}{x}} + \frac{dz}{z} = 0$$

Integrating we have,

$\log\left(\tan^{-1}\frac{y}{x}\right) + \log z = \log c$, where c is an arbitrary constant.

i.e., $z\tan^{-1}\frac{y}{x} = c$ *i.e.*, $\frac{y}{x} = \tan\left(\frac{c}{z}\right)$

i.e., $y = x\tan\left(\frac{c}{z}\right)$, which is the required general solution for c being an arbitrary constant.

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16. **Solve : $(yz + z^2)dx - xzdy + xydz = 0$** [BH 2010]

Solution Here the given differential equation is

$(yz + z^2)dx - xz\,dy + xy\,dz = 0$ ... (1)

Comparing (1) with its standard form $p\,dx + q\,dy + r\,dz = 0$

we have, $p = yz + z^2$, $q = -xz$, $r = xy$.

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= (yz + z^2)(-x - x) - xz(y - y - 2z) + xy(z + z)$

$= -2xyz - 2xz^2 + 2xz^2 + 2xyz = 0.$

Thus the condition of integrability is satisfied

We see that the given differential equation is a homogeneous in form.

So to solve, we put $x = uz$ and $y = vz$

Then we have $dx = u\,dz + z\,du$ and $dy = v\,dz + z dv$. Now substituting these in (1) we have,

$(uz.\,z + z^2)(udz + zdu) - uz^2(vdz + zdv) + uvz^2dz = 0$

i.e., $(1 + v)(udz + zdu) - u(vdz + zdv) + uvdz = 0$ [since $z \neq 0$]

i.e., $(u + uv - uv + uv)dz + z(1 + v)du - uzdv = 0$

i.e., $u\,dz + z\,du - uz\frac{dv}{1+v} = 0$ *i.e.*, $\frac{dz}{z} + \frac{du}{u} - \frac{dv}{1+v} = 0$

Integrating we have, $\log z + \log u - \log(1 + v) = \log c$ where c is an arbitrary constant

i.e., $\frac{uz}{1+v} = c$ *i.e.*, $x = c\left(1 + \frac{y}{z}\right)$, since $v = \frac{y}{z}$

i.e., $zx = c(y + z)$, which is the required general solution for c being an arbitrary constant.

■ **Alternative Method**

$z(y + z)dx - x(z\,dy - y\,dz) = 0$

$\Rightarrow \frac{y+z}{z}.dx - x\left(\frac{zdy - ydz}{z^2}\right) = 0 \Rightarrow \left(1 + \frac{y}{z}\right).\,dx - x\,d\left(\frac{y}{z}\right) = 0 \Rightarrow \frac{dx}{x} - \frac{d\left(\frac{y}{z}\right)}{1 + \frac{y}{z}} = 0$

Now integrating, we get

$\log x - \log\left(1 + \frac{y}{z}\right) = \log c$, where c is an arbitrary constant.

$\Rightarrow \log x = \log c\left(1 + \frac{y}{z}\right) \Rightarrow x = c\left(1 + \frac{y}{z}\right) \Rightarrow xz = c(y + z).$

17. **Solve : $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$** [BH 1998, IAS 1990]

Solution Here the given differential equation is

$(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$ ... (1)

comparing (1) with its standard form $p\,dx + q\,dy + r\,dz = 0$,

we have, $p = y^2 + yz$, $q = xz + z^2$ and $r = y^2 - xy$.

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Now, $p\left(\frac{\partial q}{\partial z}-\frac{\partial r}{\partial y}\right)+q\left(\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}\right)+r\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)$

$= (y^2 + yz)(x + 2z - 2y + x) + (xz + z^2)(-y - y) + (y^2 - xy)(2y + z - z)$

$= 2y(y + z)(x + z - y) - 2yz(x + z) + 2y^2(y - x)$

$= 2y[x(y + z) + z^2 - y^2 - xz - z^2 + y^2 - xy] = 0$

Thus the condition for integrability is satisfied. since the given differential equation is in homogeneous form, then we may put $x = uz$ and $y = vz$ so that $dx = udz + zdu$ and $dy = vdz + zdv$. Now substituting all of them in (1) we have,

$(v^2z^2 + vz^2)(udz + zdu) + (uz^2 + z^2)(vdz + zdv) + (v^2z^2 - uvz^2)dz = 0$

i.e., $(v + v^2)(u\,dz + z\,du) + (u + 1)(vdz + zdv) + v(v - u)dz = 0$

i.e., $(uv + uv^2 + uv + v + v^2 - uv)dz + z(v + v^2)du + z(u + 1)dv = 0$

i.e., $v(1 + u)(1 + v)dz + zv(1 + v)du + z(1 + u)dv = 0$

i.e., $\frac{dz}{z}+\frac{du}{1+u}+\frac{dv}{v(1+v)}=0$

i.e., $\frac{dz}{z}+\frac{du}{1+u}+\frac{dv}{v}-\frac{dv}{1+v}=0$

Integrating we have,

$\log(z) + \log(1 + u) + \log(v) - \log(1 + v) = \log c$, c being an arbitrary constant.

i.e., $\frac{z(1+u)v}{1+v}=c$ i.e., $z\left(1+\frac{x}{z}\right)\frac{y}{z}=c\left(1+\frac{y}{z}\right)$ [since $u=\frac{x}{z}$ and $v=\frac{y}{z}$]

i.e., $y(x + z) = c(y + z)$, which is the required general solution for c being an arbitrary constant.

18. **Solve : $xdx + (y - z)dy = ydz$**

Solution Here the given differential equation can be written as

$xdx + (y - z)dy = ydz$

i.e., $xdx + (y - z)dy - ydz = 0$ ... (1)

Comparing (1) with its standard form $pdx + qdy + rdz = 0$,

we have, $p = x$, $q = y - z$ and $r = -y$

Now, $p\left(\frac{\partial q}{\partial z}-\frac{\partial r}{\partial y}\right)+q\left(\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}\right)+r\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)=x(-1+1)+(y-z)(0-0)-y(0-0)=0$

Thus the condition for integrability is satisfied. Therefore by inspection we can solve the given differential equation.

From (1) we have,

$xdx + ydy - zdy - ydz = 0$ i.e., $d\left(\frac{x^2}{2}\right)+d\left(\frac{y^2}{2}\right)-d(yz)=0$

Integrating we have, $\frac{x^2}{2}+\frac{y^2}{2}-yz=c$, where c is an arbitrary constant

i.e., $x^2 + y^2 = 2(c + yz)$, which is required general solution for c being an arbitrary constant.

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19. Solve : $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$

Solution Here the given differential equation is

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0 \qquad \text{... (1)}$$

Comparing (1) with its standard form $pdx + qdy + rdz = 0$.

We have, $p = y^2 + yz + z^2$, $q = z^2 + zx + x^2$ and $r = x^2 + xy + y^2$.

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= (y^2 + yz + z^2)(2z + x - x - 2y) + (z^2 + 2x + x^2)(zx + y - y - 2z) + (x^2 + xy + y^2)(2y + z - z - 2x) = 0$

Thus the condition of integrability is satisfied here. So here we solve the differential equation (1) by finding the auxiliary equation. Now eliminating p, q, r from (1) and the condition of integrability

$$p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0,$$

we have, $\dfrac{dx}{\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}} = \dfrac{dy}{\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}} = \dfrac{dz}{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}$ *i.e.*, $\dfrac{dx}{2(z-y)} = \dfrac{dy}{2(x-z)} = \dfrac{dz}{2(y-x)}$

$$\text{i.e., } \frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \qquad \text{... (2)}$$

From (2) we have, $\dfrac{dx}{z-y} = \dfrac{dy}{x-z} = \dfrac{dz}{y-x} = \dfrac{dx + dy + dz}{z - y + x - z + y - x} = \dfrac{dx + dy + dz}{0}$ ... (3)

From (3) we have, $dx + dy + dz = 0$

Integrating we have $x + y + z = u$ = constant (say)

Again from (2) we have,

$$\frac{2x\,dx}{2xz - 2xy} = \frac{2y\,dy}{2xy - 2yz} = \frac{2z\,dz}{2yz - 2xz} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{2xz - 2xy + 2xy - 2yz + 2yz - 2xz} \qquad \text{... (4)}$$

Since the denominator of last fraction of (4) is 0 so from (4) we have

$2x\,dx + 2y\,dy + 2z\,dz = 0$ *i.e.*, $d(x^2 + y^2 + z^2) = 0$

Integrating we have, $x^2 + y^2 + z^2 = v$ = constant (say)

We now determine two functions ϕ and ψ such that the equation (1) can be re-written as

$$\phi du + \psi dv = 0 \qquad \text{... (5)}$$

Using the expressions of u and v in (5), we have

$$\phi\, d(x + y + z) + \psi\, d(x^2 + y^2 + z^2) = 0$$

$$\text{i.e., } (\phi + 2x\psi)dx + (\phi + 2y\psi)dy + (\phi + 2z\psi)dz = 0 \qquad \text{... (6)}$$

Comparing (1) and (6) we have,

$$y^2 + yz + z^2 = \phi + 2x\psi \qquad \text{... (7)}$$

$$z^2 + zx + x^2 = \phi + 2z\psi \qquad \text{... (8)}$$

$$x^2 + xy + y^2 = \phi + 2z\psi \qquad \text{... (9)}$$

Subtracting (8) from (7) we have,

$$y^2 - x^2 + z(y - x) = 2(x - y)\psi$$

i.e., $-(x+y+z)(x-y) = 2(x-y)\psi$

i.e., $\psi = -\frac{x+y+z}{2} = -\frac{u}{2}$ [Since $u = x+y+z$ and assuming $x \neq y$] ... (10)

Now from (7), we have

$\phi = y^2 + yz + z^2 - 2x\psi = y^2 + yz + z^2 - 2x\cdot\left(-\frac{u}{2}\right)$ [From 10]

$= y^2 + yz + z^2 = x(x+y+z) = x^2 + y^2 + z^2 + xy + yz + zx$

$= \frac{1}{2}(x+y+z)^2 + \frac{1}{2}(x^2+y^2+z^2)$

$= \frac{1}{2}(u^2 + v)$ [since $u = x+y+z$ and $v = x^2+y^2+z^2$] ... (11)

Using (10) and (11) in (5), we have

$$\frac{1}{2}(u^2+v)du - \frac{u}{2}dv = 0 \text{ i.e., } u^2du + (v\,du - u\,dv) = 0$$

i.e., $du + \frac{vdu - udv}{u^2} = 0$ i.e., $du - d\left(\frac{v}{u}\right) = 0$ i.e., $d\left(u - \frac{v}{u}\right) = 0$

Integrating we have, $u - \frac{v}{u} = 2c$, where c is an arbitrary constant.

i.e., $u^2 - v = 2cu$ i.e., $(x+y+z)^2 - (x^2+y^2+z^2) = 2c(x+y+z)$

i.e., $2(xy+yz+zx) = 2c(x+y+z)$ i.e., $xy + yz + zx = c(x+y+z)$, which is the required general solution for c being an arbitrary constant.

20. Solve $2yzdx - 2xzdy - (x^2-y^2)(z-1)dz = 0$ [BH 1999]

Solution Here the given differential equation is

$2yzdx - 2xzdy - (x^2-y^2)(z-1)dz = 0$ ... (1)

Comparing (1) with its standard form $pdx + qdy + rdz = 0$,

we have, $p = 2yz$, $q = -2xz$, $r = -(x^2-y^2)(z-1)$

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= 2yz(-2x - 2yz + 2y) - 2xz(-2xz + 2x - 2y) - 4(x^2-y^2)(z-1)z = 0$

Therefore the condition for integrability has been satisfied.

Thus we can solve the problem by considering one variable as constant. Thus to solve, we choose z as a constant so that we have $dz = 0$. Then from (1) we have,

$2yzdx - 2xz\,dy = 0$

i.e., $y\,dx - xdy = 0$ [since z = constant $\neq 0$]

i.e., $\frac{ydx - xdy}{y^2} = 0$ [since $y \neq 0$]

i.e., $d\left(\frac{x}{y}\right) = 0$

Integrating we have, $\frac{x}{y} = f(z)$

where $f(z)$ is a function of z as z is treated as constant. ... (2)

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Now considering the differential of $\frac{x}{y} = f(z)$, we have,

$\frac{ydx - xdy}{y^2} = f'(z)dz$ i.e., $2yzdx - 2xz\,dy - 2y^2zf'(z)dz = 0$ ... (3)

Comparing the equations (1) and (3), we have

$2y^2z\,f'(z) = (x^2 - y^2)(z - 1)$ i.e., $2zf'(z) = \left(\frac{x^2}{y^2} - 1\right)(z - 1) = (z - 1)\,[(f(z))^2 - 1]$, since $\frac{x}{y} = f(z)$

i.e., $2z\frac{d\,f(z)}{dz} = (z - 1)[(f(z))^2 - 1]$ i.e., $\frac{2d\,f(z)}{(f(z))^2 - 1} = \frac{z-1}{z}dz$

i.e., $\left[\frac{1}{f(z)-1} - \frac{1}{f(z)+1}\right]df(z) = \left(1 - \frac{1}{z}\right)dz$

Integrating we have,

$\log(f(z) - 1) - \log(f(z) + 1) = z - \log z + \log c$, where c is an arbitrary constant

i.e., $\frac{f(z)-1}{f(z)+1} = \frac{ce^z}{z}$ i.e., $f(z) = \frac{z + ce^z}{z - ce^z}$

i.e., $\frac{x}{y} = \frac{z + ce^z}{z - ce^z}$ [From (2)]

i.e., $x(ce^z - z) + y(ce^z + z) = 0$

which is the required general solution for the given differential equation where c is an arbitrary constant.

■ **Alternative Method**

Since condition of integrability for the given differential equation (1) has been satisfied

i.e., $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0$

then the differential equation (1) can be solved by inspection. From (1) we have,

$2z(y\,dx - x\,dy) - (x^2 - y^2)(z - 1)dz = 0$

i.e., $\frac{ydx - xdy}{x^2 - y^2} - \frac{z-1}{2z}dz = 0$ i.e., $\frac{ydx - xdy}{y^2\left(\frac{x^2}{y^2} - 1\right)} - \frac{z-1}{2z}dz = 0$

i.e., $\frac{2d\left(\frac{x}{y}\right)}{\frac{x^2}{y^2} - 1} - \frac{z-1}{z}dz = 0$ i.e., $\frac{\left(\frac{x}{y}+1\right) - \left(\frac{x}{y}-1\right)}{\left(\frac{x}{y}+1\right)\left(\frac{x}{y}-1\right)}d\left(\frac{x}{y}\right) - \frac{z-1}{z}dz = 0$

i.e., $\frac{d\left(\frac{x}{y}\right)}{\frac{x}{y} - 1} - \frac{d\left(\frac{x}{y}\right)}{\frac{x}{y} + 1} + \frac{dz}{z} = dz$

Integrating we have, $z + c = \log\left(\frac{x}{y} - 1\right) - \log\left(\frac{x}{y} + 1\right) + \log(z)$ where c is an arbitrary constant.

i.e., $\log\frac{z(x-y)}{x+y} = z + c$

i.e., $z(x - y) = (x + y)e^{z+c}$, which is the required general solution for the given differential equation where c is an arbitrary constant.

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21. **Solve : $(1 + yz)dx + x(z - x)dy - (1 + xy)dz = 0$** [BH 2008]

Solution Here the given differential equation is

$(1 + yz)dx + x(z - x)dy - (1 + xy)dz = 0$ ... (1)

Comparing the differential equation (1) with its standard form $pdx + qdy + rdz = 0$, we have $p = 1 + yz$, $q = x(z - x)$ and $r = -(1 + xy)$

Now, $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$

$= (1 + yz)(x + x) + x(z - x)(-y - y) - (1 + xy)(z - z + zx)$

$= 2x + 2xyz - 2xyz + 2x^2y - 2x - 2x^2y = 0$

Thus the condition for integrability has been satisfied.

To solve we put x = constant.

If x = constant then $dx = 0$

and from (1) we have $x(z - x)dy - (1 + xy)dz = 0$

i.e., $\frac{dz}{z - y} - \frac{x\,dy}{1 + xy} = 0$

Integrating we have

$\log(z - x) - \log(1 + xy) = \log \phi(x)$, where $\phi(x)$ is a function of x and independent in y and z

i.e., $\frac{z - x}{1 + xy} = \phi(x)$

i.e., $z - x = (1 + xy)\,\phi(x)$ ... (2)

To determine $\phi(x)$, we substitute a constant in both (1) and (2), we put $y = 0$ in both (1) and (2) and we have,

$dx - dz = 0$ ... (3)

and $z - x = \phi(x)$ ... (4)

From (3) we have, $d(z - x) = 0$

Integrating we have, $z - x = c$, ... (5)

Comparing (4) and (5) we have,

$\phi(x) = c$ ... (6)

Using (6) in (2) we have, the required general solution for the given differential equation is as follows :

$(z - x) = c(1 + xy)$, where c is an arbitrary constant.

Therefore the required general solution is $z - x = c(1 + xy)$.

Exercise

1. Show that $(2x + y^2 + 2zx)dx + 2xydy + x^2\,dz = 0$ is integrable. [Bangalore 1995]

2. Verify the condition of integrability $yzdx = zxdy + y^2dz$.

3. Verify that $x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$ is integrable.

4. Verify the condition of integrability for $zdx + zdy + 2(x + y + \sin z)dz = 0$ [Bangalore 2004]

- **Solve the following differential equations (Q.5 to Q.24)**

5. $yz\,dx + (b - x)(z\,dy + y\,dz)$

6. $2yz\,dx + 2x\,dy - xy(1 + z)dz = 0$ [Garhwal 2010]

7. $zydx = zxdy + y^2dz$ [Mysore 2004, Delhi BA Math 2007]

8. $y\,dx + (x + 3y^2)dy + 2z\,dz = 0$

9. $(1 - yz)dx + (xz - 1)dy + y(x - y)dz = 0$

10. $yz^2(x^2 - yz)dx + zx^2(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$

11. $yzdx - xzdy - (xy + y^2)dz = 0$

12. $(x - y)dx - xdy + zdz = 0$ [Delhi Math (G) 2006]

13. $yz(1 + x)dx + zx(1 + y)dy + xy(1 + z)dz = 0$

14. $xz^3dx - zdy + 2ydz = 0$

15. $2z(y + z)dx - 2xzdy + (y^2 + z^2 - x^2 + 2yz - 2xz)dz = 0$

16. $x\,dx + (y - z)dy - y\,dz = 0$

17. $dx + dy + (x + y + z + 1)dz = 0$ [Meerut 2012]

18. $(y + z)dx + (z + x)dy + (x + y)dz = 0$

19. $yz(1 + x)dx - 2xzdy + xydz = 0$

20. $xydz + z\log z(ydx - x\,dy) = 0$

21. $(y - z)(y + z - 2x)dx + (z - x)(z + x - 2y)dy + (x - y)(x + y - 2z)dz = 0$

22. $(yz + z^2)dx - xzdy + xydz = 0$ [T.H. 2008]

23. $(2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$ [CU 2002, Meerut 2012]
[*Hint* : Put $x = uz$, $y = vz$]

24. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$ [Delhi Math 1998]

Answers

5. $yz = c(b - x)$
6. $x^2y = cze^z$
7. $z = ce^{\frac{x}{y}}$
8. $xy + y^3 + z^2 = c$
9. $yz - 1 = c(x - y)$
10. $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = c$
11. $yz = c(x + y)$
12. $x^2 - 2xy + z^2 = c$
13. $xyz = ce^{-(x+y+z)}$
14. $x^2z^2 - 2y = cz^2$
15. $x(1 - cz) = (y + z)(1 + cz)$
16. $x^2 + y^2 - 2yz = c$
17. $(x + y + z).\, e^z = c.$
18. $xy + yz + zx = c$
19. $xe^x = c\cdot y^2z$
20. $x\log z = cy$
21. $x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2) = c$
22. $xz = c(y + z)$
23. $x^2 - xy + y^2 = cz$
24. $xy - x^2 - y^2 = cz$

[c is an arbitrary constant for Q. No. 5 to 24]

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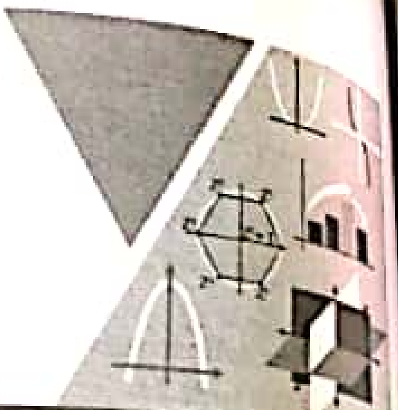
13 CHAPTER



Solution of a Differential Equation by Reducing its Order

Introduction

In this chapter we discuss the methods for solving ordinary differential equation of higher order containing non-linear form by reducing the order of the differential equation. We consider a differential equation of one independent variable x and one dependent variable y. However, the differential equation may be of two types namely the differential equation is not directly involved with dependent variables and in second type where the differential equation is not directly involved with independent variable x. Below we discuss the solution procedures for those two types of differential equations.

- **Type-1 : The differential equation does not contain the dependent variable y directly :** In this type the form of the differential equation is as follows :

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, x\right) = 0 \qquad \ldots (1)$$

To solve (1) we first reduce the order of the differential equation is (1) by substituting $p = \frac{dy}{dx}$. Then we have $\frac{dp}{dx} = \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}} = \frac{d^{n-2}p}{dx^{n-2}}, \frac{d^n y}{dx^n} = \frac{d^{n-1}p}{dx^{n-1}}$ thus the differential equation (1) reduces to

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, \ldots, \frac{dp}{dx}, p, x\right) = 0 \qquad \ldots (2)$$

Equation (2) may easily be solved for p and let after solution, we have $p = \phi_1(x)$ from which we have, $\frac{dy}{dx} = \phi_1(x)$ and then integrating we can get required general solution of the differential equation (1).

Notes

1. If the differential equation is of the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, x\right) = 0 \qquad \ldots (3)$$

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then to solve (3) we have to substitute $p = \frac{d^{n-1}y}{dx^{n-1}}$, so that, $\frac{dp}{dx} = \frac{d^n y}{dx^n}$ and (3) reduces to

$$f\left(\frac{dp}{dx}, p, x\right) = 0 \quad \text{... (4)}$$

Clearly equation (4) is a first order differential equation on p which can be solved easily. Let the solution of (4) be $p = \phi_2(x)$ which gives $\frac{d^{n-1}y}{dx^{n-1}} = \phi_2(x)$ and integrating $(n-1)$ times, we get the required general solution for the differential equation (3).

2. If the differential equation is of the following form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-2}y}{dx^{n-2}}, x\right) = 0 \quad \text{... (5)}$$

[*i.e.*, the differential equation containing only two differential operators $\frac{d^{n-2}y}{dx^{n-2}}$ and $\frac{d^n y}{dx^n}$ such that the order of those two differential operators differ by two]

In this case to solve (5), we have to substitute $p = \frac{d^{n-2}y}{dx^{n-2}}$ so that we have $\frac{d^2 p}{dx^2} = \frac{d^n y}{dx^n}$

and with these transformations (5) reduces as $f\left(\frac{d^2 p}{dx^2}, p, x\right) = 0$ ... (6)

Clearly (6) is a second order differential equation and whose solution can easily be obtained let the solution of the differential equation (6) be

$$p = \frac{d^{n-2}y}{dx^{n-2}} = \phi_3(x) \quad \text{... (7)}$$

Then integrating $(n-2)$ times (7), successively we have the required general solution for the differential equation of the form (5).

3. If in the form (1) the lowest order differential term is of r-th order, *i.e.*, if the differential equation (1) is of the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{d^{r+1}y}{dx^{r+1}}, \frac{d^r y}{dx^r}, x\right) = 0 \quad \text{... (8)}$$

Then to solve (8) substitute $p = \frac{d^r y}{dx^r}$ and then proceeding in the similar way, as described above, we get the required general solution of (8).

◆ **Type-2 : The differential equation which does not contain the independent variable x directly :**

In this type the form of the differential equation is :

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y\right) = 0 \quad \text{... (9)}$$

To solve (9) we substitute $p = \frac{dy}{dx}$. Then we have, $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$ and so on. Thus after these substitutions, the differential equation (9) becomes a differential equation which

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is of p and y only. So it can be solved easily and let after solution, we have,

$$p = \frac{dy}{dx} = \phi_4(y)$$

Then integrating $\frac{dy}{\phi_4(y)} = dx$ we get the required general solution of the differential equation (9).

Note

If in the form (9) the lowest order differential term is of r-th order, *i.e.*, if the differential equation (9) is of the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{d^{r+1}y}{dx^{r+1}}, \frac{d^r y}{dx^r}, y\right) = 0 \qquad \ldots (10)$$

Then to solve (10) we have to substitute $p = \frac{d^r y}{dx^r}$ and proceeding as similar way we have,

$$\frac{d^r y}{dx^r} = \phi_5(y) \qquad \ldots (11)$$

If $r = 1$, then (11) can be solved as follows

$\frac{dy}{dx} = \phi_5(y)$ giving $\frac{dy}{\phi_5(y)} = dx$ and after integrating we may get the general solutions. If $r = 2$, then (11) reduces as

$$\frac{d^2 y}{dx^2} = \phi_5(y) \qquad \ldots (12)$$

To solve (12) we multiply both sides of (12) by $2\frac{dy}{dx}$ and then we have,

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\phi_5(y)\frac{dy}{dx} \text{ i.e., } \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\phi_5(y)\frac{dy}{dx} \text{ giving } d\left(\frac{dy}{dx}\right)^2 = 2\phi_5(y)dy$$

Integrating we have, $\left(\frac{dy}{dx}\right)^2 = 2\int\phi_5(y)dy + A$, A being an arbitrary constant

$$\text{i.e., } dx = \pm\frac{dy}{\sqrt{A + 2\int\phi_5(y)dy}}$$

After integrating above equation, we get the required general solution of (10).

Similarly one can obtain the solution of the equation on (11) for $r = 3, 4, \ldots$

Below in illustrative examples, some problems are to be solved (for both Type 1 and Type 2) by reducing the order of the corresponding differential equation.

Illustrative Examples

1. **Solve : $x\frac{d^2y}{dx^2} = 1$**

Solution Here the given differential equation is $x\frac{d^2y}{dx^2} = 1 \qquad \ldots (1)$

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To solve the differential equation (1) by reducing its order

Substitute $\frac{dy}{dx} = p \quad \therefore \frac{d^2y}{dx^2} = \frac{dp}{dx}$

So the given differential equation (1) becomes $x\frac{dp}{dx} = 1$ *i.e.*, $dp = \frac{dx}{x}$

Now integrating, we have

$p = \log x + A$ *i.e.*, $\frac{dy}{dx} = \log x + A$ *i.e.*, $dy = \log x dx + A dx$, where A is an arbitrary constant.

Again integrating, we have

$y = x\log x - x + Ax + B$, which is the required general solution, where A and B are two arbitrary constants.

2. **Solve :** $\frac{d^2y}{dx^2} = \cos x$

Solution Here the given differential equation is $\frac{d^2y}{dx^2} = \cos x$ ... (1)

We solve the differential equation by reducing its order.

We substitute $\frac{dy}{dx} = p$ then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$,

So, the given differential equation becomes

$\frac{dp}{dx} = \cos x$ *i.e.*, $dp = \cos x \cdot dx$

Now integrating, we have

$p = \sin x + A$, where A is an arbitrary constant.

i.e., $\frac{dy}{dx} = \sin x + A$ *i.e.*, $dy = \sin x - dx + Adx$

Again integrating, we have

$y = -\cos x + Ax + B$, which is the required general solution, where A and B are two arbitrary constants.

Note

The given differential equation can easily be solved by integrating twice.

3. **Solve :** $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$ [CP 1989, KH 2002]

Solution Here the given differential equation is $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$ ... (1)

We solve the differential equation (1) by reducing its order.

Therefore to solve (1), we substitute $p = \frac{dy}{dx}$.

Then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$. Using these substitutions, the differential equation (1) reduces to

$$(1 - x^2)\frac{dp}{dx} - xp = 2 \quad \text{... (2)}$$

Clearly the equation (2) is a first order differential equation on p and hence it can be solved easily.

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From (2) we have, $\frac{dp}{dx}-\frac{x}{1-x^2}p=\frac{2}{1-x^2}$ ... (3)

Clearly differential equation (3) is a linear differential equation. So its integrating factor (I.F.) is

$$e^{-\int\frac{x}{1-x^2}dx}=e^{-\frac{1}{2}\int\frac{2x}{1-x^2}dx}=\sqrt{1-x^2}$$

Multiplying both sides of (3) by I.F. = $\sqrt{1-x^2}$ and then integrating we have,

$p\sqrt{1-x^2}=\int\frac{2}{\sqrt{1-x^2}}dx+A$, where A is an arbitrary constant.

i.e., $p=\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}(2\sin^{-1}x+A)$

i.e., $dy=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}dx+\frac{A\,dx}{\sqrt{1-x^2}}$ i.e., $dy=d((\sin^{-1}x)^2+A\sin^{-1}x)$

Integrating we have,

$y=(\sin^{-1}x)^2+A\sin^{-1}x+B$, which is the required general solution for A and B being two arbitrary constants.

4. Solve : $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0$ [BH 1998]

Solution Here the given differential equation is

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0 \quad ...(1)$$

To solve (1) by reducing its order, we substitute, $p=\frac{dy}{dx}$. Then $\frac{d^2y}{dx^2}=\frac{dp}{dx}$.

Using above two substitutions, the equation (1) reduces to

$$\frac{dp}{dx}+p^2+1=0 \quad ...(2)$$

(2) is a first order differential equation on p and x and hence it can be solved easily.

We have from (2), $\frac{dp}{1+p^2}+dx=0$

Integrating we have, $x+\tan^{-1}p=A$, where A is an arbitrary constant.

i.e., $\tan^{-1}p=A-x$ i.e., $p=\frac{dy}{dx}=\tan(A-x)$, solving $dy+\tan(x-A)dx=0$

Integrating we have, $y=\log\cos(x-A)+B$, which is the required general solution of the given differential equation for A and B being two arbitrary constants.

5. Solve : $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0$

Solution Here the given differential equation is

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0 \quad ...(1)$$

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The equation (1) can be solved by reducing its order. We substitute $p = \frac{dy}{dx}$, then we have $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ with these transformations (1) reduces to

$$x\frac{dp}{dx} + p + 1 = 0 \quad \text{... (2)}$$

Clearly (2) is a first order differential equation on p and x.

Then we have from (2),

$$xdp + pdx + dx = 0 \quad i.e., \; d(px + x) = 0$$

Integrating we have,

$px + x = A$, where A is an arbitrary constant

$$i.e., \; p = \frac{dy}{dx} = \frac{A - x}{x} \quad i.e., \; dy = \left(\frac{A}{x} - 1\right)dx$$

Integrating we have, $y = A \log x - x + B$, where B is an arbitrary constant.

i.e., $x + y = A \log x + B$, which is the required general solution for A and B being two arbitrary constants.

[CH 1989]

6. **Solve :** $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Solution Here the given differential equation is

$$\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{... (1)}$$

To solve (1), we substitute $p = \frac{dy}{dx}$. Then we have, $\frac{d^2y}{dx^2} = \frac{dp}{dx}$.

With these substitutions, (1) reduces to : $\frac{dp}{dx} = \sqrt{1 + p^2}$ ... (2)

Clearly equation (2) is a first order differential equation and it can be solved easily.

We have from (2), $\frac{dp}{\sqrt{1 + p^2}} = dx$

Integrating we have,

$\sinh^{-1} p = x + A$, where A is an arbitrary constant.

$$i.e., \; p = \frac{dy}{dx} = \sinh(x + A) \quad i.e., \; dy = \sinh(x + A)dx$$

Integrating we have, $y = \cosh(x + A) + B$, which is the required general solution of the given differential equation for A and B being two arbitrary constants.

7. **Solve :** $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2x = 0$

Solution Here the given differential equation is

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2x = 0 \quad \text{... (1)}$$

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Equation (1) is a second order differential equation and it can be solved after reducing its order. We substitute $p = \frac{dy}{dx}$, so that we have, $\frac{dp}{dx} = \frac{d^2y}{dx^2}$. With these transformations (1) reduces to

$$\frac{dp}{dx} + xp + 2x = 0$$

i.e., $\frac{dp}{dx} + x(p + 2) = 0$ ... (2)

Equation (2) is a first order differential equation on p and x. So we have from (2),

$$\frac{dp}{p+2} + x\,dx = 0.$$

Integrating we have, $\log(p + 2) + \frac{x^2}{2} = A$, where A is an arbitrary constant.

i.e., $p + 2 = e^{A-\frac{x^2}{2}} = A_1 e^{-\frac{x^2}{2}}$, where $A_1 = e^A$.

i.e., $p = \frac{dy}{dx} = A_1 e^{-\frac{x^2}{2}} - 2$ i.e. $dy = A_1 e^{-\frac{x^2}{2}} dx - 2dx$

Integrating we have,

$y = B - 2x + A_1 \int e^{-\frac{x^2}{2}} dx$, which is the required general solution for A_1 and B being two arbitrary constants.

8. Solve : $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 0$

Solution Here the given differential equation is

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 0 \quad \text{... (1)}$$

To solve (1), we substitute $p = \frac{dy}{dx}$. Then we have $\frac{dp}{dx} = \frac{d^2y}{dx^2}$ with these transformations, the equation (1) reduces to

$$\frac{dp}{dx} + p + p^2 = 0 \quad \text{... (2)}$$

Equation (2) is a first order differential equation on p and so we have from (2),

$\frac{dp}{p+p^2} + dx = 0$ i.e., $\frac{(p+1)-p}{p(p+1)}dp + dx = 0$ i.e., $\frac{dp}{p} - \frac{dp}{p+1} + dx = 0$

Integrating we have, $\log p - \log(p + 1) + x + \log A = 0$, where A is an arbitrary constant.

i.e., $\frac{p+1}{p} = Ae^x$ i.e., $1 + \frac{1}{p} = Ae^x$ i.e., $\frac{1}{p} = Ae^x - 1$ i.e., $p = \frac{dy}{dx} = \frac{1}{Ae^x - 1}$

i.e., $dy = \frac{dy}{Ae^x - 1} = \frac{e^{-x}dx}{A - e^{-x}} = d(\log(A - e^{-x}))$

Integrating we get,

$y = \log(A - e^{-x}) + B$, which is the required general solution for the given differential equation for A and B being two arbitrary constants.

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9. Solve : $(1+x^2)\frac{d^2y}{dx^2}+1+\left(\frac{dy}{dx}\right)^2=0$

Solution Here the given differential equation is

$$(1+x^2)\frac{d^2y}{dx^2}+1+\left(\frac{dy}{dx}\right)^2=0 \quad \text{... (1)}$$

Clearly the differential equation (1) is a second order non-linear differential equation and it can be solved by reducing its order. So to solve, we put $p=\frac{dy}{dx}$.

Then $\frac{dp}{dx}=\frac{d^2y}{dx^2}$. With these substitutions (1) reduces to

$$(1+x^2)\frac{dp}{dx}+1+p^2=0 \quad \text{... (2)}$$

Equation (2) is a first order differential equation, so we have form (2),

$$\frac{dp}{1+p^2}+\frac{dx}{1+x^2}=0$$

Integrating we have, $\tan^{-1}p+\tan^{-1}x=\tan^{-1}A$, where A is an arbitrary constant.

i.e., $\tan^{-1}p=\tan^{-1}A-\tan^{-1}x=\tan^{-1}\frac{A-x}{1+Ax}$

i.e., $p=\frac{dy}{dx}=\frac{A-x}{1+Ax}$ *i.e.*, $dy=\frac{A-x}{1+Ax}dx=\frac{Adx}{1+Ax}-\frac{1}{A}\frac{Ax+1-1}{Ax+1}dx$

i.e., $dy=\frac{Adx}{1+Ax}-\frac{1}{A}dx+\frac{1}{A^2}\frac{Adx}{Ax+1}$ *i.e.*, $dy=\left(1+\frac{1}{A^2}\right)\frac{Adx}{Ax+1}-\frac{1}{A}dx$

Integrating we have,

$y=\left(1+\frac{1}{A^2}\right)\log(Ax+1)-\frac{x}{A}+B$, which is the required general solution for the given differential equation for A and B being two arbitrary constants.

10. Solve : $\frac{d^2y}{dx^2}+3\frac{dy}{dx}=2e^x+1$

Solution Here the given differential equation is

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}=2e^x+1 \quad \text{... (1)}$$

Differential equation (1) can be solved by reducing its order

For this purpose we substitute, $p=\frac{dy}{dx}$. Then $\frac{dp}{dx}=\frac{d^2y}{dx^2}$

with these transformations, equation (1) reduces to

$$\frac{dp}{dx}+3p=2e^x+1 \quad \text{... (2)}$$

Clearly equation (2) is a first order linear differential equation.

Therefore to solve (2), the I.F. is $e^{\int 3dx}=e^{3x}$.

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Now multiplying (2) by the I.F. e^{3x} and then integrating we have,

$pe^{3x} = 2\int e^{4x}dx + \int e^{3x}dx + A$, where A is an arbitrary constant

$$= \frac{1}{2}e^{4x} + \frac{e^{3x}}{3} + A$$

i.e., $p = \frac{dy}{dx} = \frac{1}{2}e^{x} + \frac{1}{3} + Ae^{-3x}$ *i.e.*, $dy = \frac{1}{2}e^{x}\,dx + \frac{1}{3}dx + Ae^{-3x}dx$

Integrating we have,

$y + B = \frac{1}{2}e^{x} + \frac{x}{3} - \frac{A}{3}e^{-3x}$, which is the required general solution of the given differential equation for A and B being two arbitrary constants.

11. Solve : $\frac{d^2y}{dx^2}\frac{dy}{dx} + 1 = x$

Solution Here the given differential equation is $\frac{d^2y}{dx^2}\cdot\frac{dy}{dx} + 1 = x$ ... (1)

To solve (1) we first reduce the order of the equation (1) by substituting $p = \frac{dy}{dx}$.

Then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$

With these substitution (1) reduces to $p\frac{dp}{dx} + 1 = x$ ... (2)

Clearly equation (2) is a first order differential equation. We have from equation (2)

$p\frac{dp}{dx} = x - 1$ *i.e.*, $p\,dp = (x-1)dx$

Integrating we get,

$\frac{p^2}{2} = \frac{x^2}{2} - x + \frac{1}{2} + \frac{A^2}{2}$, where A is an arbitrary constant

i.e., $p^2 = (x-1)^2 + A^2$ *i.e.*, $p = \frac{dy}{dx} = \pm\sqrt{(x-1)^2 + A^2}$

Integrating twice (3) with respect to x, we have

$y = \pm\frac{x\sqrt{(x-1)^2 + A^2}}{2} \pm \frac{A^2}{2}\sinh^{-1}\frac{x-A}{A} + B$,

which is the required general solution for A and B being two arbitrary constants.

12. Solve : $\frac{d^4y}{dx^4} + \tan x\frac{d^3y}{dx^3} = 0$

Solution Here the given differential equation is $\frac{d^4y}{dx^4} + \tan x\frac{d^3y}{dx^3} = 0$ ... (1)

The differential equation (1) can be solved by reducing its order. To solve (1), we substitute $p = \frac{d^3y}{dx^3}$. Then $\frac{d^4y}{dx^4} = \frac{dp}{dx}$ with these substitutions the equation (1) reduces to

$\frac{dp}{dx} + p\tan x = 0$ ... (2)

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Equation (2) is a first order differential equation. So we have from (2),

$\frac{dp}{p} + \tan x \, dx = 0$

Integrating we have, $\log p + \log \sec x = \log A$; where A is an arbitrary constant.

i.e., $p \sec x = A$ i.e., $p = \frac{d^3y}{dx^3} = A \cos x$ ... (3)

Integrating (3) thrice successively we have

$\frac{d^2y}{dx^2} = A \sin x + B$, where B is an arbitrary constant.

$\frac{dy}{dx} = -A \cos x + Bx + C$, where C is an arbitrary constant.

$y = -A \sin x + \frac{B}{2}x^2 + Cx + D$, which is the required general solution for A, B, C and D are arbitrary constants.

13. Solve : $\frac{d^4y}{dx^4} + a^2\frac{d^2y}{dx^2} = e^x$

Solution Here the given differential equation is $\frac{d^4y}{dx^4} + a^2\frac{d^2y}{dx^2} = e^x$ ... (1)

The differential equation (1) can be solved by reducing its order.

To solve (1), we substitute, $p = \frac{d^2y}{dx^2}$. Then $\frac{d^4y}{dx^4} = \frac{d^2p}{dx^2}$ with these substitutions (1) reduces to ... (2)

$\frac{d^2p}{dx^2} + a^2p = e^x$

Clearly equation (2) is a second order differential equation on p.

So to solve (2), we first find its reduced differential equation as follows ... (3)

$\frac{d^2p}{dx^2} + a^2p = 0$

Let $p = ce^{mx}$ $(c \neq 0)$ be a trial solution of (3). Then the auxiliary equation of (3) is as follows

$m^2 + a^2 = 0$ giving $m = \pm ia$.

Thus the complementary function of (2) becomes

$p_C = A \cos ax + B \sin ax$, where A and B are two arbitrary constants.

To find particular integral (P.I.) of (2), we use the differential operator $D = \frac{d}{dx}$ and we have from (2), the P.I.

$p_p = \frac{1}{D^2 + a^2}(e^x) = \frac{1}{1+a^2}e^x$

Therefore the general solution of (2) becomes

$p = p_c + p_p = A \cos ax + B \sin ax + \frac{1}{1+a^2}e^x$ ... (4)

i.e., $\frac{d^2y}{dx^2} = A \cos ax + B \sin ax + \frac{1}{1+a^2}e^x$

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Integrating (4) successively with respect to x twice we have,

$$\frac{dy}{dx} = \frac{1}{a}\,[A \sin ax - B \cos ax] + \frac{1}{1+a^2}e^x + C, \text{ where C is an arbitrary constant.}$$

$$y = -\frac{1}{a^2}\,[A \cos ax + B \sin ax] + \frac{1}{1+a^2}e^x + Cx + D,$$

which is the required general solution for A, B, C and D are arbitrary constants.

14. Solve : $\frac{d^2y}{dx^2} = \frac{k}{y^3}$

Solution Here the given differential equation is $\frac{d^2y}{dx^2} = \frac{k}{y^3}$ ... (1)

We solve the differential equation (1) by reducing its order. We put $\frac{dy}{dx} = p$

So, $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$.

Thus the given differential equation (1) becomes

$$p\frac{dp}{dy} = \frac{k}{y^3} \Rightarrow pdp = \frac{k}{y^3}dy$$

Now integrating, we get

$$\frac{p^2}{2} = -\frac{k}{2y^2} + \frac{A}{2}, \text{ where A is an arbitrary constant.}$$

i.e., $p^2 = A - \frac{k}{y^2}$ i.e., $p = \pm\frac{\sqrt{Ay^2 - k}}{y}$ i.e., $\frac{dy}{dx} = \pm\frac{\sqrt{Ay^2 - k}}{y}$ i.e., $\frac{y\,dy}{\sqrt{Ay^2 - k}} = \pm\, dx$

Again integrating,

$$\frac{\sqrt{Ay^2 - k}}{A} = \pm x + B, \text{ where B is an arbitrary constant.}$$

i.e., $\sqrt{Ay^2 - k} = \pm Ax + C$, where $C = AB$

i.e., $Ay^2 = k + (C \pm Ax)^2$, where A and C are two arbitrary constants.

Which is the required general solution.

15. Solve : $\frac{d^2y}{dx^2} = \tan y \sec^2 y$, given $\frac{dy}{dx} = 0$ when $y = 0$

Solution Here the given differential equation is $\frac{d^2y}{dx^2} = \tan y \sec^2 y$

We solve the differential equation (1) by reducing its order. ... (1)

We put $\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$.

Thus the given differential equation reduces to

$$p\frac{dp}{dy} = \tan y \sec^2 y \quad i.e., \quad pdp = \tan y \sec^2 y\, dy$$

Now integrating we get

$\frac{p^2}{2} = \frac{\tan^2 y}{2} + \frac{A}{2}$, where A is an arbitrary constant.

i.e., $p^2 = A + \tan^2 y$ ... (2)

Accordingly $\frac{dy}{dx} = p = 0$ where $y = 0$

So, from (2), A = 0. Thus the equation (2) reduces to :

$p^2 = \tan^2 y$, giving $(p - \tan y)(p + \tan y) = 0$

either $p - \tan y = 0$ ... (3)

or $p + \tan y = 0$ ... (4)

From (3) $\frac{dy}{dx} = \tan y$, giving $\cot y\, dy = dx$

Now integrating, we get, $\log \sin y = x + \log B_1$, where B_1 is an arbitrary constant.

i.e., $\sin y - B_1 e^x = 0$ ... (5)

From (4) $\frac{dy}{dx} = -\tan y$, giving $\cot y\, dy = -dx$

integrating, we get $\log \sin y = -x + \log B_2$, where B_2 is an arbitrary constant.

i.e., $\sin y - B_2 e^{-x} = 0$ ... (6)

From (5) & (6), the general solution of equation (1) is $(\sin y - Be^x)(\sin y - Be^{-x}) = 0$, where B is an arbitrary constant.

Note

The given differential equation can be solved after twice successive integration.

16. Solve : $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$

Solution Here the given differential equation is $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$ ... (1)

The differential equation (1) can be solved by reducing its order. To solve (1), we substitute $p = \frac{dy}{dx}$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$. With these substitution, (1) reduces to

$y\frac{dp}{dx} + p^2 = 1$ i.e., $y\frac{dp}{dy}\frac{dy}{dx} = 1 - p^2$ i.e. $py\frac{dp}{dy} = 1 - p^2$ ... (2)

Equation (2) is a first order differential equation and we have from (2)

$y\frac{dp}{dy} = \frac{1-p^2}{p}$ i.e., $\frac{2p\,dp}{p^2-1} + \frac{2dy}{y} = 0$

Integrating we have,

$\log(p^2 - 1) + \log(y^2) = \log A$, where A is an arbitrary constant.

i.e., $(p^2 - 1)y^2 = A$ i.e., $p^2 = \frac{A}{y^2} + 1$ i.e., $p = \frac{dy}{dx} = \pm\frac{\sqrt{A + y^2}}{y}$

i.e., $\frac{y\,dy}{\sqrt{A+y^2}} = \pm dx$ ... (3)

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To evaluate (3), we put $u^2 = A + y^2$ so that we have $u\,du = y\,dy$. Then from (3) have,

$$\frac{u\,du}{u} = \pm\,dx \quad i.e.,\ du = \pm\,dx$$

Integrating we have, $u = B \pm x$, where B is an arbitrary constant.

$$i.e.,\ \sqrt{A + y^2} = B \pm x \quad i.e.,\ A + y^2 = (B \pm x)^2$$

i.e., $y^2 = (B \pm x)^2 - A$, which is the required general solution for A and B being two arbitrary constants.

17. **Solve :** $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2 \log y$ **[BH 1991, CH 1995, KH 2000]**

Solution Here the given differential equation is $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2 \log y$ ... (1)

The differential equation (1) can be solved by reducing its order.

To solve (1) we substitute $p = \dfrac{dy}{dx}$. Then $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$

With these transformations (1) reduces to

$$y\frac{dp}{dx} - p^2 = y^2 \log y \quad i.e.,\ y\frac{dp}{dy}\frac{dy}{dx} - p^2 = y^2 \log y$$

$$i.e.,\ py\frac{dp}{dy} - p^2 = y^2 \log y \quad \text{... (2)}$$

Clearly (2) is a first order differential equation. To solve (2) we put $u = p^2$.

Then $\dfrac{du}{dy} = 2p\dfrac{dp}{dy}$. Thus we have from (2),

$$\frac{y}{2}\frac{du}{dy} - u = y^2 \log y \quad i.e.,\ \frac{du}{dy} - \frac{2}{y}u = 2y \log y \quad \text{... (3)}$$

Clearly equation (3) is a linear differential equation.

So I.F. of (3) $= e^{\int -\frac{2}{y}dy} = e^{-2\log y} = \dfrac{1}{y^2}$

Multiplying (3) with the I.F. $\dfrac{1}{y^2}$ and then integrating we have,

$\dfrac{u}{y^2} = 2\displaystyle\int \frac{\log y}{y}\,dy + A$, where A is an arbitrary constant.

$$i.e.,\ \frac{p^2}{y^2} = (\log y)^2 + A \quad i.e.,\ p^2 = y^2[A + (\log y)^2] \quad i.e.,\ p = y\sqrt{A + (\log y)^2}$$

$$i.e.,\ \frac{dy}{dx} = y\sqrt{A + (\log y)^2} \quad i.e.,\ \frac{dy}{y\sqrt{A + (\log y)^2}} = dx$$

Integrating both sides we have,

$\displaystyle\int \frac{dy}{y\sqrt{A + (\log y)^2}} = \int dx + B$, where B, is another arbitrary constant ... (4)

To evaluate (4), we substitute, $t^2 = A + (\log y)^2$

Then $2t\,dt = 2\log y\,\frac{dy}{y}$ i.e., $\frac{dy}{y} = \frac{t\,dt}{\log y} = \frac{t\,dt}{\sqrt{t^2 - A}}$

Using these transformation in (4), we have after integration,

$\int \frac{t\,dt}{\sqrt{t^2 - A}t} = \int dx + B$ i.e., $x + B = \int \frac{dt}{\sqrt{t^2 - A}} = \cosh^{-1}\frac{t}{\sqrt{A}}$ i.e., $t = \sqrt{A}\cosh(x + B)$

i.e., $t^2 = A\cosh^2(x + B)$ i.e., $A + (\log y)^2 = A\cosh^2(x + B)$

i.e., $(\log y)^2 = A\{\cosh^2(x + B) - 1\} = A\sinh^2(x + B)$

i.e., $\log y = \sqrt{A}\sinh(x + B)$, which is the required general solution for the given differential equation where A and B are two arbitrary constants.

18. **Solve :** $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y^2$

Solution Here the given differential equation is $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y^2$ ... (1)

We solve (1) by reducing its order. Let us substitute $p = \frac{dy}{dx}$.

Then we have, $\frac{d^2y}{dx^2} = \frac{dp}{dx}$. With these transformations, (1) reduces to ... (2)

$y\frac{dp}{dx} + p^2 = y^2$ i.e., $y\frac{dp}{dy}\frac{dy}{dx} + p^2 = y^2$ i.e., $py\frac{dp}{dy} + p^2 = y^2$

Equation (2) is a first order differential equation and to solve (2), we put $u = p^2$.

Then $du = 2p\,dp$, with these transformation (2) reduces to : ... (3)

$\frac{1}{2}y\frac{du}{dy} + u = y^2$ i.e., $\frac{du}{dy} + \frac{2u}{y} = 2y$

Clearly (3) is a linear differential equation. So its I.F. is $e^{\int \frac{2}{y}dy} = 2^{2\log y} = y^2$.

Multiplying (3) by the I.F. y^2 and then integrating we have,

$uy^2 = \int 2y^3 dy + A$, where A is an arbitrary constant.

i.e., $uy^2 = \frac{y^4}{2} + A$ i.e., $u = \frac{y^2}{2} + \frac{A}{y^2}$ i.e., $p^2 = \frac{y^2}{2} + \frac{A}{y^2} = \frac{y^4 + 2A}{2y^2}$ ... (4)

i.e., $p = \frac{dy}{dx} = \frac{\sqrt{2A + y^4}}{\sqrt{2}y}$

To evaluate (4), we substitute, $t^2 = 2A + y^4$

Then $2t\,dt = 4y^3dy$ i.e., $\frac{dy}{y} = \frac{t\,dt}{2y^4} = \frac{t\,dt}{2(t^2 - 2A)}$

Using these substitutions in (4) we have,

$y\,dy = \frac{t\,dt}{2y^2} = \frac{t\,dt}{\sqrt{t^2 - 2A}}$

$\frac{t}{\sqrt{t^2 - 2A}}\frac{dt}{dx} = \frac{t}{\sqrt{2}}$ i.e., $dx = \frac{\sqrt{2}}{\sqrt{t^2 - 2A}}dt$

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Integrating we have,

$x + B\sqrt{2} = \sqrt{2}\cosh^{-1}\frac{t}{\sqrt{2A}}$, where B is an arbitrary constant

i.e., $B + \frac{x}{\sqrt{2}} = \cosh^{-1}\frac{t}{\sqrt{2A}}$ *i.e.*, $\frac{t}{\sqrt{2A}} = \cosh\left(B + \frac{x}{\sqrt{2}}\right)$

i.e., $\sqrt{2A + y^4} = \sqrt{2A}\cosh\left(B + \frac{x}{\sqrt{2}}\right)$

i.e., $2A + y^4 = 2A\cosh^2\left(B + \frac{x}{\sqrt{2}}\right)$

i.e., $y^4 = 2A\left\{\cosh^2\left(B + \frac{x}{\sqrt{2}}\right) - 1\right\} = 2A\sinh^2\left(B + \frac{x}{\sqrt{2}}\right)$

i.e., $y^2 = \sqrt{2A}\sinh\left(B + \frac{x}{\sqrt{2}}\right)$,

which is the required general solution for A and B being two arbitrary constants.

Exercise

- **Solve the following differential equations by reducing its order :**

1. $\frac{d^2y}{dx^2} = c$
2. $\cos^2 x\frac{d^2y}{dx^2} = 1$
3. $\operatorname{cosec} x\frac{d^2y}{dx^2} = 1$
4. $\frac{d^2y}{dx^2} = e^{2x}$
5. $\frac{d^2x}{dt^2} = \sec^2 t$
6. $\frac{d^2x}{dt^2} = \frac{1}{\sqrt{x}}$
7. $\frac{d^2y}{dx^2} = xe^x$
8. $\frac{d^2y}{dx^2} - \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 2 = 0$
9. $\frac{d^3y}{dx^3}\cdot\frac{d^2y}{dx^2} - 2 = 0$
10. $\frac{d^5y}{dx^5} - a^2\frac{d^3y}{dx^3} = e^{bx}$
11. $\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0$
12. $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$
13. $\frac{1 + x^2}{2}\cdot\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 1 = 0$
14. $x^2\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 0$
15. $\left\{\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2}\right\}^2 - a^2\left\{\left(\frac{dy}{dx}\right)^2 + b^2\left(\frac{d^2y}{dx^2}\right)^2\right\} = 0$

    [*Hint* : Put $p = \frac{dy}{dx}$]
16. $\frac{d^2x}{dt^2} = \frac{1}{\sqrt{1 - t^2}}$ given, $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$
17. $\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$

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Answers

1. $y = \frac{c}{2}x^2 + Ax + B$
2. $y = \log \sec x + Ax + B$
3. $y = Ax + B - \sin x$
4. $y = \frac{1}{4}e^{2x} + Ax + B$
5. $x = \log \sec t + Ax + B$
6. $3t = 2(\sqrt{x} - 2A)\left(\sqrt{x} + A\right)^{\frac{1}{2}} + B$
7. $y = (x - 2)e^x + Ax + B$
8. $y = 2x - 2\log(1 - Ae^{2x}) + B$
9. $y = \frac{8}{15}(x + A)^{\frac{5}{2}} + Bx + C$
10. $y = Ae^{ax} + Be^{-ax} + \frac{e^{bx}}{b^3(b^2 - a^2)} + C + Dx + Ex^2$
11. $\sin(y + A) = e^{x+B}$
12. $y = (\sin^{-1}x)^2 + A\sin^{-1}x + B$
13. $y = \log(1 + x^2) + A\tan^{-1}x + B$
14. $y = Bx^{\frac{5}{2}}\cos\left(\frac{\sqrt{3}}{2}\log Ax\right) + Cx + D$
15. $(Ay - Be^{Ax})^2 = a^2(1 + b^2A^2)$
16. $x = t\sin^{-1}t + t + \sqrt{1-t^2} + 1$
17. $e^y(Ax + B) = 1$

[In all cases A, B, C, D, E are arbitrary constants.]

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14

CHAPTER

Introduction and Formulation of Partial Differential Equation

Introduction

A differential equation containing one or more partial derivatives is known as partial differential equation. In mathematics and allied science partial differential equations have wide applications. Both in pure and applied mathematics, many engineering branches, physical, chemical and biological sciences have required various types of partial differential equation and its solution. In this chapter we will discuss types of partial differential equation and its formulation.

Definition : A differential equation containing at least one partial derivative of an unknown function of two or more independent variables is known as partial differential equation.

Example-1 : Let z be a function of two independent variables x and y.

Then $\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}=5$, $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial y^2}=a^2$ are the examples of partial differential equations.

Let us provide example of some partial differential equations.

(i) $\frac{\partial z}{\partial x}+4\frac{\partial z}{\partial y}=x+z$,

(ii) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x^2+y^2z$

(iii) $\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial x^2}=x^2$,

(iv) $\frac{\partial^2 z}{\partial x\,\partial y}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

(v) $\left(\frac{\partial z}{\partial x}\right)^2+x^2\left(\frac{\partial^2 z}{\partial x^2}\right)^3=\frac{\partial^3 z}{\partial^2 x\,\partial y}$,

(vi) $\frac{\partial z}{\partial x}=\left(\frac{\partial^2 z}{\partial x^2}\right)^{\frac{5}{2}}+\left(\frac{\partial^2 z}{\partial y^2}\right)^{\frac{5}{2}}$

14.1 Order of a Partial Differential Equation

The order of the highest order partial derivatives involving in a partial differential equation is known its order. For example in the above examples, order of first and second are one, order of third, fourth and sixth differential equations are two where as order of fifth differential equation is three.

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14.2 Degree of a Partial Differential Equation

After rationalizing a partial differential equation, the degree of highest order partial derivative is the order of the corresponding partial differential equation. For example the degree of first four differential equations are one, degree of fifth differential equation is also one (since the power of $\frac{\partial^3 z}{\partial^2 x \partial y}$ is one) but the degree of sixth differential equation is five, which can be obtained after rationalizing it. However it should be remembered that order of each differential equation exists but degree may not. For example for the partial differential equation, $\sin\frac{\partial z}{\partial x} + \tan\frac{\partial^2 z}{\partial x^2} = x^2 \log\frac{\partial^2 z}{\partial y^2}$, there is no degree although its order is two.

14.3 Types of Partial Differential Equation

Below we define different types of first order partial differential equation. Let z be a dependent variable and it is a function of two independent variables x and y. Then a first order partial differential equation should be a function of the dependent variable z, independent variables x and y and the two first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. We generally denote the two first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively by p and q. So let the required function be $F(x, y, z, p, q)$.

14.4 Linear Equation

If first order partial differential equation $F(x, y, z, p, q) = 0$ can be represented as $a(x, y)p + b(x, y)q = A(x, y) + B(x, y)z$ [where $a(x, y)$, $b(x, y)$, $A(x, y)$, $B(x, y)$ are functions of x and y or constants], then the differential equation is said to be first order linear partial differential equation.

- **Example :** $p + q = 1$ and $px + qy = 2x + 3z$ are two examples of linear first order partial differential equation.

14.5 Semi-linear Partial Differential Equation

If a first order partial differential equation $F(x, y, z, p, q) = 0$ can be represented as $a(x, y)p + b(x, y)q = c(x, y, z)$ [where a and b are functions of x and y where as c is a function of x, y and z] then this type of partial differential equation is known as semi-linear partial differential equation. For example $2p + 3q = xyz$, $px + qy = x^2y^2z^3$ are the examples of semi-linear partial differential equation.

14.6 Quasi-linear Partial Differential Equation

If a first order partial differential equation $F(x, y, z, p, q) = 0$ can be represented as $a(x, y, z)p + b(x, y, z)q = c(x, y, z)$ [where a, b and c all are functions of x, y and z only], then this

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type of partial differential equation is known as quasi-linear partial differential equation. For example, the partial differential equations of the form, $x^2yz^2p + xq = y^2z^3$, $xzp + zq = x$ are the examples of quasi-linear partial differential equation.

Notes

1. It should be noted that all other type partial differential equations, except linear, semi-linear and quasi-linear are non-linear differential equations. For example $x^2y^2p^2 + xyq^2 = z^2$, $xp^3 + yq = 1$, etc. are the examples of non-linear partial differential equation.
2. We have already noticed that if $z \equiv z(x, y)$ then the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ can be represented by p and q respectively. On the other hand if $u \equiv u(x, y)$, then the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can respectively be represented as u_x and u_y.

14.7 Formation of Partial Differential Equation

In this section, we derive the procedure to determine or formulate a partial differential equation from a given functional relation.

◆ **Type-1** : Let us consider a function containing two arbitrary constants a, b as $F(x, y, z, a, b) = 0$, where z, is a function of x and y. ... (1)

Now differentiating (1) partially with respect to x and y respectively we have,

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \qquad \text{... (2)}$$

and $$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \qquad \text{... (3)}$$

Now (2) and (3) both containing the arbitrary constants a and b. So eliminating a and b among (1), (2) and (3), we have required differential equation as follows :

$$\phi(x, y, z, p, q) = 0 \qquad \text{... (4)}$$

Below we discuss three possible types of examples to derive partial differential equation from a functional relation.

Type-1.1 : Let the number of arbitrary constants be less than number of independent variables. In this case, we may formulate a partial differential equation of first order, however, it may not be unique.

Illustration-1 : Consider a functional relation $z = ax + y$, where z is a function of two independent variables x and y and a be an arbitrary constant. To formulate a partial differential equation from the relation $z = ax + y$, let us differentiate partially with respect to x and y, then we have, $\frac{\partial z}{\partial x} = p = a$ and $\frac{\partial z}{\partial y} = q = 1$. Thus the required partial differential equation is either $z = px + y$ or $q = 1$.

Type-1.2 : Let the number of arbitrary constants be same as number of independent variable. In this case we have a unique partial differential equation of order one which can be formulated by eliminating those arbitrary constants.

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Illustration-2 : Formulate a partial differential equation from $z = ax - by$ by eliminating arbitrary constants a and b.

Here the given functional relation is $z = ax - by$ ... (5)

To formulate a partial differential equation by eliminating two arbitrary constants a and b, differentiating partially (5) with respect to x and y we have respectively,

$$\frac{\partial z}{\partial x} = p = a \text{ and } \frac{\partial z}{\partial y} = q = b$$

Substituting these values in (5), we have,

$z = px - qy$, which is the required partial differential equation, and of course, this is a unique partial differential equation of order one.

Type-1.3 : Let the number of arbitrary constants be greater than number of independent variables. In this case, the partial differential equation, obtained by eliminating those arbitrary constants usually be of greater than order one.

Illustration-3 : Formulate a partial differential equation by eliminating arbitrary constants a, b, c from $z = ax + by + c(x^2 + y^2)$

Here the given functional relation is ... (6)

$$z = ax + by + c(x^2 + y^2)$$

Differentiating (6) partially with respect to x and y respectively we have ... (7)

$$\frac{\partial z}{\partial x} = p = a + 2cx \quad \text{... (8)}$$

$$\frac{\partial z}{\partial y} = q = b + 2cy$$

Again differentiating (7) partially with respect to x we have, ... (9)

$$\frac{\partial^2 z}{\partial x^2} = 2c \text{ giving } c = \frac{1}{2}\frac{\partial^2 z}{\partial x^2}$$

Using the value of c (from (9)) in (7) and (8) we have, $a = p - x\frac{\partial^2 z}{\partial x^2}$ and $b = q - y\frac{\partial^2 z}{\partial x^2}$

Substituting a, b, c in (6), we have, $z = \left(p - x\frac{d^2 z}{dx^2}\right)x + \left(q - y\frac{d^2 z}{dx^2}\right)y + \frac{1}{2}\frac{d^2 z}{dx^2}$, which is the required partial differential equation.

Note

It should be noted that the formulated partial differential equation in **illustration-3** is not unique, since c can be obtained from differentiating (8) partially with respect to y, and then the required partial differential equation becomes, $z = \left(p - x\frac{d^2 z}{dy^2}\right)x + \left(q - y\frac{d^2 z}{dy^2}\right)y + \frac{1}{2}\frac{d^2 z}{dy^2}$.

- **Type-2 :** Formulation of a partial differential equation by eliminating one or more arbitrary functions. In this case also we have several sub-cases which are illustrated below :

Type-2.1 : Consider a functional relation between the dependent variable z and the independent variables x and y with one function which is as follows : ... (10)

$$z = f(x, y)$$

Then the required partial differential equation must be of first order [since there is only one function in the functional relation 10] and it can be obtained by eliminating f from (10) and two partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Illustration-4 : Obtain a partial differential equation from $z = f(x + 2y)$ by eliminating the function f.

Here the given functional relation is $z = f(x + 2y)$ ... (11)

Now differentiating (11) partially with respect to x and y respectively we have,

$$\frac{\partial z}{\partial x} = p = f'(x + 2y)$$

and $$\frac{\partial z}{\partial y} = q = 2f'(x + 2y) \quad \text{... (12)}$$

From (12) and (13) we have, $2p = q$, which is the required partial differential equation ... (13)

Type-2.2 : Consider a functional relation between the dependent variable z and the independent variables x and y with more than one function. In this situation the order of the formulated partial differential equation will be same as number of arbitrary functions.

Illustration-5 : Formulate a partial differential equation from $z = f(x + y^2) + g(x + y)$ by eliminating two arbitrary functions f and g.

Here the given functional relation is $z = f(x + y^2) + g(x + y)$

Differentiating (1) partially with respect to x and y respectively we have, ... (14)

$$\frac{\partial z}{\partial x} = f'(x + y^2) + g'(x + y)$$

and $$\frac{\partial z}{\partial y} = 2yf'(x + y^2) + g'(x + y) \quad \text{... (15)}$$

Again differentiating (15) and (16) partially with respect to x and y respectively we have, ... (16)

$$\frac{\partial^2 z}{\partial x^2} = f''(x + y^2) + g''(x + y)$$

$$\frac{\partial^2 z}{\partial y^2} = 4y^2 f''(x + y^2) = 2f'(x + y^2) + g''(x + y) \quad \text{... (17)}$$

Now subtracting (15) from (16) we have, ... (18)

$$(2y - 1)f'(x + y^2) = \frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} \quad i.e., f'(x + y^2) = \frac{1}{2y-1}\left(\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\right)$$

Again subtracting (17) from (18) we have, ... (19)

$$\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = (4y^2 - 1)f''(x + y^2) + 2f'(x + y^2) = (4y^2 - 1)f''(x + y^2) + \frac{2}{2y-1}\left(\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\right) \quad \text{... (20)}$$

Again differentiating (15) partially with respect to y we have,

$$\frac{\partial^2 z}{\partial x \partial y} = 2yf''(x + y^2) + g''(x + y)$$

... (21)

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Now subtracting (17) from (20) we have,

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial x^2} = (2y-1)f''(x+y^2) \text{ i.e., } f''(x+y^2) = \frac{1}{2y-1}\left(\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial x^2}\right) \quad \dots (22)$$

Using (22) in (20) we have,

$$\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = (4y^2-1)\frac{1}{2y-1}\left(\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial x^2}\right) + \frac{2}{2y-1}\left(\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\right)$$

$$\text{i.e., } (2y-1)\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) = (4y^2-1)\left(\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial x^2}\right) + 2\left(\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\right)$$

which is the required partial differential equation.

- **Type-3 :** Formulation of a partial differential equation by eliminating an arbitrary function f which is connected by the relation $f(u, v) = 0$ where $u = u(x, y, z)$ and $v = v(x, y, z)$ are known functions of x, y and z.

In this case taking partial derivatives of $f(u, v) = 0$ with respect to x and y respectively we have,

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \dots (23)$$

and $$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \dots (24)$$

Since u and v are known function of x, y and z then $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ all are known and hence the required partial differential equation can be obtained by eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (23) and (24) which is given by $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$. This type of problem is illustrated in worked out example 21.

Illustrative Examples

1. **Form partial differential equation by eliminating arbitrary constants a and b from the relation $z = (x + a)(y + b)$**

Solution Given relation is $z = (x+a)(y+b)$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = 1\cdot(y+b) \Rightarrow p = y + b \quad \dots (2)$$

and $$\frac{\partial z}{\partial y} = 1(x+a) \Rightarrow q = x + a \quad \dots (3)$$

Now, we get from (1) by using (2) and (3) we get, $z = pq$

i.e., $z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$ which is the required partial differential equation.

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2. **Form a partial differential equation from $z = a(x + y) + b$ by eliminating the arbitrary constants a and b.**

Solution Here the given relation is $z = a(x + y) + b$ ... (1)

Now differentiating (1) partially with respect to x and y, we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = a$

Therefore $p = a$ ... (2)

and $q = a$ ... (3)

From (2) and (3) we have $p = q$

i.e., $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ which is the required partial differential equation.

3. **Formulate a partial differential equation from $z = (x - a)^2 + (y - b)^2$ by eliminating the arbitrary constants a and b.** [VH 2007]

Solution Here the given relation is $z = (x - a)^2 + (y - b)^2$ ... (1)

Differentiating (1) partially with respect to x and y respectively we have,

$\frac{\partial z}{\partial x} = p = 2(x - a)$ giving $\frac{p}{2} = x - a$ ... (2)

and $\frac{\partial z}{\partial y} = q = 2(y - b)$ giving $\frac{q}{2} = y - b$ ... (3)

Substituting the values of p and q from (2) and (3) in (1) we have,

$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$ *i.e.*, $p^2 + q^2 = 4z$, which is the required partial differential equation.

4. **Formulate partial differential equation from $z = ax + (1 - a)y + b$ by eliminating the arbitrary constants a and b.** [NBH 2006]

Solution Here the given relation is $z = ax + (1 - a)y + b$ ... (1)

Differentiating (1) partially with respect to x and y respectively we have,

$\frac{\partial z}{\partial x} = p = a$ and $\frac{\partial z}{\partial y} = q = 1 - a = 1 - p$

i.e., $p + q = 1$, which is the required partial differential equation.

5. **Formulate a partial differential equation from $az + b = a^2x + y$ by eliminating the arbitrary constants a and b.** [BH 2004]

Solution Here the given relation is $az + b = a^2x + y$ ... (1)

Differentiating (1) partially with respect to x and y respectively we have,

$a\frac{\partial z}{\partial x} = ap = a^2$ giving $p = a$ ... (2)

and $a\frac{\partial z}{\partial y} = aq = 1$ giving $q = \frac{1}{a}$ ... (3)

From (2) and (3) we have, $pq = 1$, which is the required partial differential equation.

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6. Find the differential equation from the equation $z = (x^2 + a)(y^2 + b)$ by eliminating the arbitrary constants a and b. [IAS 1997]

Solution Here the given relation is $z = (x^2 + a)(y^2 + b)$ ... (1)

Differentiating (1) partially with respect to x and y respectively we have,

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \text{ giving } y^2 + b = \frac{p}{2x} \quad \text{... (2)}$$

and $$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \text{ giving } x^2 + a = \frac{q}{2y} \quad \text{... (3)}$$

Using the expressions of (2) and (3) in (1), we have,

$$z = \frac{pq}{(2x)(2y)}$$ i.e., $pq = 4xyz$, which is the required partial differential equation.

7. From a partial differential equation by eliminating the arbitrary constants h and k from $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$ [IAS 1996]

Similar Problem **: Find the differential equation of all spheres of radius λ, having centre in the xy-plane.**

Solution Here the given relation is $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$ ... (1)

Differentiating (1) partially with respect to x and y respectively, we have,

$$2(x - h) + 2z\frac{\partial z}{\partial x} = 0, \text{ i.e., } zp + (x - h) = 0 \text{ i.e., } x - h = -pz \quad \text{... (2)}$$

Also $$2(y - k) + 2z\frac{\partial z}{\partial x} = 0 \text{ i.e., } zq + (y - k) = 0 \text{ i.e., } y - k = -zq \quad \text{... (3)}$$

Using (2) and (3) in (1) we have,

$$(-pz)^2 + (-qz)^2 + z^2 = \lambda^2$$

i.e., $z^2(1 + p^2 + q^2) = \lambda^2$, which is the equation of required partial differential equation.

8. Eliminate the arbitrary constants a and b from $z = ae^{bx}\cos by$ to form a partial differential equation.

Solution Here the given relation is $z = ae^{bx}\cos by$ ... (1)

Differentiating (1) with respect to x and y partially respectively, we have

$$\frac{\partial z}{\partial x} = abe^{bx}\cos by \quad \text{... (2)} \qquad \frac{\partial z}{\partial y} = -abe^{bx}\sin by \quad \text{... (3)}$$

Again differentiating (2) and (3) with respect to x and y partially respectively we have,

$$\frac{\partial^2 z}{\partial x^2} = ab^2e^{bx}\cos bx \quad \text{... (4)} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = -ab^2e^{bx}\cos by \quad \text{... (5)}$$

Adding (4) and (5) we have,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$ which is the required partial differential equation.

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9. **Form partial differential equation by eliminating arbitrary constants a and b from the relation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.**

Solution Given relation is $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ ... (1)

Differentiating (1) partially with respect to x and y respectively, we have

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow p = \frac{x}{a^2} \text{ giving } a^2 = \frac{x}{p}$$

and $2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow q = \frac{y}{b^2}$ giving $b^2 = \frac{y}{q}$

Now from (1), $2z = px + qy$ which is the required partial differential equation.

10. **Find the differential equation of the set of all right circular cones whose axes coincide with z-axis.** [IAS 1998]

Solution From geometry, we know that the equation of the set of all right circular cones whose axes is z-axis is $x^2 + y^2 = (z - c)^2 \tan^2\alpha$ ... (1)

where α is the semi-vertical angle and $(0, 0, c)$ be the vertex. Thus here c and α are two arbitrary constants. Now differentiating (1) partially with respect to x and y respectively we have,

$2x = \{2(z - c)\tan^2\alpha\}\frac{\partial z}{\partial x}$ *i.e.*, $(z - c)\tan^2\alpha = \frac{x}{p}$ ... (2)

and $2y = \{2(z - c)\tan^2\alpha\}\frac{\partial z}{\partial y}$, *i.e.*, $(z - c)\tan^2\alpha = \frac{y}{q}$ ... (3)

Comparing (2) and (3) we have,

$\frac{x}{p} = \frac{y}{q}$ *i.e.*, $py = qx$, which is the equation of required partial differential equation.

11. **Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$** [IAS 2009]

Solution From Geometry, the general equation of any cone whose vertex is at the origin in given by

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, where $a, b, c\ f, g$ and h are parameters. ... (1)

Now differentiating (1) with respect to x and y partially respectively we have

$2ax + 2gz + 2hy + (2cz + 2fy + 2gx)p = 0$ *i.e.*, $p = -\frac{ax + hy + gz}{cz + fy + gx}$ ... (2)

Again, $2by + 2fz + 2hx + (2cz + 2fy + 2gx)q = 0$ *i.e.*, $q = -\frac{by + fz + hx}{cz + fy + gx}$ ... (3)

Now we have, $px + qy = -\frac{(ax^2 + hxy + gxz) + (by^2 + fyz + hxy)}{cz + fy + gx}$ [From (2) and (3)]

$$= -\frac{ax^2 + by^2 + 2hxy + fyz + gxz}{cz + fy + gx} = -\frac{(-cz^2 - fyz - gzx)}{cz + fy + gx} \text{ [From (1)]}$$

$= z$

Hence the required differential equation is $px + qy = z$ **[Proved]**

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12. Formulate a partial differential equation by eliminating the function f from $z = xf\left(\frac{y}{x}\right)$

[BH 2001]

Solution Here the given functional relation is $z = xf\left(\frac{y}{x}\right)$. ... (1)

Differentiating (1) partially with respect to x we have

$$\frac{\partial z}{\partial x} = p = f\left(\frac{y}{x}\right) - x\cdot\frac{y}{x^2}f'\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'(x)$$

$$\text{i.e., } xp = xf\left(\frac{y}{x}\right) - yf'(x) = z - yf'(x) \quad \text{... (2)}$$

$$\text{Also } \frac{\partial z}{\partial x} = q = x.\frac{1}{x}f'\left(\frac{y}{x}\right) \text{ i.e., } q = f'\left(\frac{y}{x}\right) \quad \text{... (3)}$$

Using (3) in (2) we have, $px = z - qy$

i.e., $px + qy = z$, which is the equation of required partial differential equation.

13. Formulate a partial differential equation by eliminating the function ϕ from the relation $lx + my + nz = \phi(x^2 + y^2 + z^2)$

[NBH 2007]

Solution Here the given relation is $lx + my + nz = \phi(x^2 + y^2 + z^2)$ ... (1)

Differentiating (1) partially with respect to x and y respectively we have,

$$l + n\frac{\partial z}{\partial x} = \left(2x + 2z\frac{\partial z}{\partial x}\right)\phi'(x^2 + y^2 + z^2) \text{ i.e., } \frac{l + np}{2(x + pz)} = \phi'(x^2 + y^2 + z^2) \quad \text{... (2)}$$

$$\text{and } m + n\frac{\partial z}{\partial y} = \left(2y + 2z\frac{\partial z}{\partial y}\right)\phi'(x^2 + y^2 + z^2) \text{ i.e., } \frac{m + nq}{2(y + qz)} = \phi'(x^2 + y^2 + z^2) \quad \text{... (3)}$$

Comparing (2) and (3) we have, $\frac{l + np}{2(x + pz)} = \frac{m + nq}{2(y + qz)}$

i.e., $(l + nq)(y + qz) = (m + nq)(x + pz)$ which is the required partial differential equation.

14. Obtain a partial differential equation from the relation $z = \phi(x + iy) + \psi(x - iy)$

[NBH 2008]

Solution Here the given relation is $z = \phi(x + iy) + \psi(x - iy)$ ... (1)

Differentiating (1) partially with respect to x twice we have,

$$\frac{\partial z}{\partial x} = \phi'(x + iy) + \psi'(x - iy) \quad \text{... (2)}$$

$$\text{and } \frac{\partial^2 z}{\partial x^2} = \phi''(x + iy) + \psi''(x - iy) \quad \text{... (3)}$$

Again differentiating (1) partially with respect to y twice we have,

$$\frac{\partial z}{\partial y} = i\phi'(x + iy) - i\psi'(x - iy) \quad \text{... (4)}$$

$$\text{and } \frac{\partial^2 z}{\partial y^2} = i^2\phi''(x + iy) + i^2\psi''(x - iy) = -[\phi''(x + iy) + \psi''(x + iy)] \quad \text{... (5)}$$

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Adding (3) and (5) we have,

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0,$$ which is the required partial differential equation.

15. **Obtain a partial differential equation from $z = f(x^2 - y) + g(x^2 + y)$ by eliminating the arbitrary functions f and g** [IAS 1996]

Solution Here the given relation is $z = f(x^2 - y) + g(x^2 + y)$ ... (1)

Differenting (1) with respect to x and y partially we have,

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2xg'(x^2 + y) = 2x[f'(x^2 - y) + g'(x^2 + y)]$$ ... (2)

and $$\frac{\partial z}{\partial y} = -f'(x^2 - y) + g'(x^2 + y)$$ ... (3)

Again differentiating partially (2) and (3) with respect to x and y respectively we have,

$$\frac{\partial^2 z}{\partial x^2} = 2x\{2xf''(x^2 - y) + 2xg''(x^2 + y)\} + 2\{f'(x^2 - y) + g'(x^2 + y)\}$$

$$= 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} + \frac{1}{x}\frac{\partial z}{\partial x} \quad \text{[From (2)]}$$ ... (4)

and $$\frac{\partial^2 z}{\partial y^2} = f''(x^2 - y) + g''(x^2 + y)$$ ... (5)

Using (5) in (4) we have, $$\frac{\partial^2 z}{\partial x^2} = 4x^2\frac{\partial^2 z}{\partial y^2} + \frac{1}{x}\frac{\partial z}{\partial x}$$

i.e., $$x\frac{\partial^2 z}{\partial x^2} - 4x^3\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial x} = 0,$$ which is the required partial differential equation.

16. **Find a partial differential equation from $z = f(x - y) + g(x + y)$ by eliminating the arbitrary functions f and g.** [BH 2004]

Solution Here the given relation is $z = f(x - y) + g(x + y)$ ... (1)

Differentiating (1) partially with respect to x twice we have

$$\frac{\partial z}{\partial x} = f'(x - y) + g'(x + y) \quad \text{... (2)} \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} = f''(x - y) + g''(x + y)$$ ... (3)

Again differentiating (1) partially with respect to y twice we have

$$\frac{\partial z}{\partial y} = -f'(x - y) + g'(x + y) \quad \text{... (4)} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = f''(x - y) + g''(x + y)$$ ... (5)

Comparing (3) and (5) we have,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} \quad i.e., \quad \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0,$$ which is the required partial differential equation.

17. **Form a partial differential equation from $u = f\left(\sqrt{x^2 + y^2}\right)$ by eliminating the arbitrary function f.** [BH 2005]

Solution Here the given relation is $u = f\left(\sqrt{x^2 + y^2}\right)$ ... (1)

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Differentiating (1) partially with respect to x and y respectively we have,

$$\frac{\partial u}{\partial x} = f'\left(\sqrt{x^2+y^2}\right)\cdot\frac{x}{\sqrt{x^2+y^2}} \quad \text{i.e., } f'\left(\sqrt{x^2+y^2}\right) = \frac{\sqrt{x^2+y^2}}{x}\frac{\partial u}{\partial x} \qquad \dots (2)$$

and $$\frac{\partial u}{\partial y} = f'\left(\sqrt{x^2+y^2}\right)\cdot\frac{y}{\sqrt{x^2+y^2}} \quad \text{i.e., } f'\left(\sqrt{x^2+y^2}\right) = \frac{\sqrt{x^2+y^2}}{y}\frac{\partial u}{\partial y} \qquad \dots (3)$$

Comparing (2) and (3) we have, $\frac{\sqrt{x^2+y^2}}{x}\frac{\partial u}{\partial x} = \frac{\sqrt{x^2+y^2}}{y}\frac{\partial u}{\partial y}$

i.e., $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$, which is the required partial differential equation.

Note

In last problem we see that u is the dependent variable instead of z. However the differential equation $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$ can be represented as $yu_x - xu_y = 0$.

18. **Eliminating the arbitrary function f from the relation $z = e^x f(x + y)$, formulate a partial differential equation.**

Solution Here the given relation is $z = e^x f(x + y)$ ... (1)

differentiating (1) partially with respect to x and y we have respectively,

$$\frac{\partial z}{\partial x} = p = e^x f'(x+y) + e^x f(x+y) \qquad \dots (2)$$

and $$\frac{\partial z}{\partial y} = q = e^x f'(x+y) \qquad \dots (3)$$

Using (1) and (3) in (2) we have,

$p = q + z$, which is the required partial differential equation.

19. **Formulate a partial differential equation by eliminating ϕ and ψ from $z = y\,\phi(x) + x\,\psi(y)$**

Solution Here the given relation is $z = y\phi(x) + x\psi(y)$ ... (1)

Differentiating (1) partially with respect x and y respectively we have,

$$\frac{\partial z}{\partial x} = y\,\phi'(x) + \psi(y) \qquad \dots (2)$$

$$\frac{\partial z}{\partial y} = \phi(x) + x\,\psi'(y) \qquad \dots (3)$$

Again differentiating (2) with respect to y partially we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \phi'(x) + \psi'(y) \qquad \dots (4)$$

Now from (2) we have, $\phi'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - \psi(y)\right]$

Again from (3) we have, $\psi'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - \phi(x)\right]$

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Using above (2) in (4) we have,

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - \psi(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - \phi(x)\right]$$

i.e., $xy\dfrac{\partial^2 z}{\partial x\,\partial y} = x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} - (y\phi(x) + x\psi(y))$

i.e., $xy\dfrac{\partial^2 z}{\partial x\,\partial y} = x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} - z$ [From (1)], which is the required partial differential equation.

20. **Find a partial differential equation by eliminating the arbitrary functions f and g from $z = f(x + ay) + g(x - ay)$** [BH 2010]

Solution Here the given relation is $z = f(x + ay) + g(x - ay)$ ... (1)

Differentiating (1) partially with respect to x and y respectively we have,

$$\frac{\partial z}{\partial x} = f'(x + ay) + g'(x - ay) \quad \text{... (2)}$$

and $\dfrac{\partial z}{\partial y} = af'(x + ay) - ag'(x - ay)$ ... (3)

Again differentiating (2) partially with respect to x and (3) partially with respect to y we have,

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ay) + g''(x - ay)$$

and $\dfrac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2 g''(x - ay) = a^2[f''(x + ay) + g''(x - ay)]$

i.e., $\dfrac{\partial^2 z}{\partial y^2} = a^2\dfrac{\partial^2 z}{\partial x^2}$, which is the required partial differential equation.

21. **Formulate a partial differential equation from the relation $f\left(\dfrac{x-a}{z-c}, \dfrac{y-b}{z-c}\right) = 0$.**

Solution Here the given functional relation is $f\left(\dfrac{x-a}{z-c}, \dfrac{y-b}{z-c}\right) = 0$ ... (1)

Let $u = \dfrac{x-a}{z-c}$ ... (2) and $v = \dfrac{y-b}{z-c}$ ... (3)

Then we have from (1), $f(u, v) = 0$ ... (4)

Differentiating (4) with respect to x and y respectively we have

$\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} = 0$ ... (4) and $\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} = 0$ ... (5)

From (2) and (3) we have, $\dfrac{\partial u}{\partial x} = \dfrac{1}{z-c} - \dfrac{(x-a)}{(z-c)^2}\dfrac{\partial z}{\partial x} = \dfrac{1}{z-c} - \dfrac{x-a}{(z-c)^2}p$,

$\dfrac{\partial u}{\partial y} = -\dfrac{(x-a)}{(z-c)^2}\dfrac{\partial z}{\partial y} = -\dfrac{x-a}{(z-c)^2}q$, $\dfrac{\partial v}{\partial x} = -\dfrac{(y-b)}{(z-c)^2}\dfrac{\partial z}{\partial x} = -\dfrac{(y-b)}{(z-c)^2}p$

and $\dfrac{\partial v}{\partial y} = \dfrac{1}{z-c} - \dfrac{y-b}{(z-c)^2}\dfrac{\partial z}{\partial y} = \dfrac{1}{z-c} - \dfrac{y-b}{(z-c)^2}q$

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Using above expression in (4) and (5) we respectively have,

$$\left\{\frac{1}{z-c}-\frac{x-a}{(z-c)^2}p\right\}\frac{\partial f}{\partial u}+\left\{-\frac{y-b}{(z-c)^2}p\right\}\frac{\partial f}{\partial v}=0 \quad \text{... (6)}$$

and $$\left\{-\frac{x-a}{(z-c)^2}q\right\}\frac{\partial f}{\partial u}+\left\{\frac{1}{z-c}-\frac{y-b}{(z-c)^2}q\right\}\frac{\partial f}{\partial v}=0 \quad \text{... (7)}$$

Since $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are not trivially equivalent to zero, so we have from (6) and (7),

$$\begin{vmatrix} \frac{1}{z-c}-\frac{x-a}{(z-c)^2}p & -\frac{y-b}{(z-c)^2}p \\ -\frac{x-a}{(z-c)^2}q & \frac{1}{z-c}-\frac{y-b}{(z-c)^2}q \end{vmatrix}=0$$

i.e., $\frac{1}{(z-c)^2}[\{(z-c)-(x-a)p\}\{(z-c)-(y-b)q\}-(x-a)(y-b)pq]=0$

i.e., $(z-c)\{(z-c)-(x-a)p-(y-b)q\}+(x-a)(y-b)pq-(x-a)(y-b)pq=0$

i.e., $z-c=(x-a)p+(y-b)q$, which is the required partial differential equation.

22. **Obtain a partial differential equation by eliminating the arbitrary functions f and g from $z=f(xy)+g(x/y)$.**

Solution Here the given functional relation is $z=f(xy)+g(x/y)$ ... (1)

Differentiating (1) partially with respect to x and y respectively we have,

$$\frac{\partial z}{\partial x}=y\,f'(xy)+\frac{1}{y}g'(x/y) \quad \text{... (2)}$$

and $$\frac{\partial z}{\partial y}=x\,f'(xy)-\frac{x}{y^2}g'(x/y) \quad \text{... (3)}$$

Again differentiating (2) partially with respect to x and (3) with respect to y partially we have

$$\frac{\partial^2 z}{\partial x^2}=y^2 f''(xy)+\frac{1}{y^2}g''(x/y) \quad \text{... (4)}$$

and $$\frac{\partial^2 z}{\partial y^2}=x^2 f''(xy)+\frac{x^2}{y^4}g''(x/y)+\frac{2x}{y^3}g'(x/y) \quad \text{... (5)}$$

Multiplying (4) by x^2 and (5) by y^2 and then subtracting we have,

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=-\frac{2x}{y}g'(x/y) \quad \text{... (6)}$$

Again multiplying (3) by y and (2) by x and then subtracting we have,

$$y\frac{\partial z}{\partial y}-x\frac{\partial z}{\partial x}=-\frac{2x}{y}g'(x/y) \quad \text{... (7)}$$

Comparing (6) and (7) we have,

$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=y\frac{\partial z}{\partial y}-x\frac{\partial z}{\partial x}$, which is the required partial differential equation.

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Exercise

- **Formulate partial differential equations by eliminating arbitrary constants or arbitrary functions (1-14) :**

1. $z = ax + by + a^2b^2$
2. $z = ax^3 + by^3$
3. $z = ax + by + a^3 + b^3$
4. $ax^2 + by^2 + z^2 = 1$
5. $2z = (ax + y)^2 + b$
6. $a^2x^2 + b^2y^2 + c^2z^2 = 1$
7. $z = ae^{bx} \sin by$
8. $z = f(y^2 - x^2)$
9. $z = x^m f\left(\frac{y}{x}\right)$
10. $z = e^{ax} f(x + y)$
11. $z = \frac{1}{xy} f(x + y + z)$
12. $z = f_1(xy) + f_2(x/y)$
13. $z = f_1(y + ax) + f_2(y + bx),\ a \neq b$
14. $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$
15. Find the partial differential equation of planes having equal intercepts along x axis and y axis.
16. Find the partial differential equation of all spheres whose centres are on the z-axis.
17. Find the differential equation of all surfaces of revolution having z-axis as the axis of rotation. [IAS 1997]
[Hint : Equation of any surface of revolution having z-axis as the axis of revolution is : $z = f\left(\sqrt{x^2+y^2}\right)$, where f is an arbitrary function : Then required partial differential equation is $py - qx = 0$].
18. Obtain the differential equation of the surface which are the envelope of a one parameter family of planes. [IAS 1995]
19. Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.

Answers

1. $z = px + qy + p^2q^2$
2. $3z = px + qy$
3. $z = px + qy + p^3 + q^3$
4. $z[z - px - qy] = 1$
5. $px + qy = q^2$
6. $z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 0$
7. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$
8. $py + qx = 0$
9. $px + qy = mz^2$
10. $az = p - q$
11. $z(x - y) + px(z - y) + qy(x - z) = 0$
12. $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = yq - xp$
13. $\frac{\partial^2 z}{\partial x^2} - (a + b)\frac{\partial^2 z}{\partial x \partial y} + ab\frac{\partial^2 z}{\partial y^2} = 0$
14. $(p - q)z + x - y = 0$
15. $p - q = 0$
16. $py - qx = 0$
17. $py - qx = 0$
18. $py - qx = 0$
19. $z = px + qy + a\left(p^2 + q^2 + 1\right)^{-\frac{1}{2}}$

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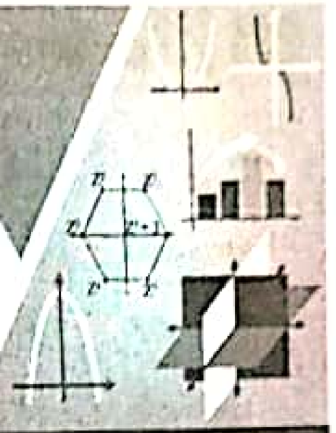
15 CHAPTER

Solution of First Order Partial Differential Equation : Lagrange's Method



Introduction

Suppose there is a first order partial differential equation of the form $Pp + Qq = R$, where p and q are two partial derivatives representing $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ with $z \equiv z(x, y)$ and P, Q, R are three functions on the dependent variable z and the independent variables x and y. Now to solve this type of partial differential equation, Lagrange's method is very helpful. We illustrate this method in next section.

15.1 Solution Technique of $Pp + Qq = R$ by Lagrange's Method

We consider a first order partial differential equation as follows :

$$Pp + Qq = R \quad \text{... (1)}$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ and P, Q, R are functions of x, y and z.

The general solution of the first order partial differential equation (1) is given by

$$f(u, v) = 0 \quad \text{... (2)}$$

which can also represented by $u = f(v)$, where f is an arbitrary function of two variables $u(x, y, z)$ and $v(x, y, z)$ in such a way that $u \equiv u(x, y, z) = A$ and $v \equiv v(x, y, z) = B$ are two independent solutions of the Lagrange's auxiliary equation given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{... (3)}$$

where A and B are two arbitrary constants and P, Q, R all three can be obtained from the partial differential equation (1). The Lagrange's auxiliary equation (3), is sometimes known as Lagrange's Subsidiary equation also.

The procedure to obtain the Lagranges' differential equation (1) is discussed below :

Differentiating (2) with respect to x and y partially we have,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \quad \text{... (4)}$$

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and $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$ ... (5)

Now eliminating two partial derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ between (4) and (5) we have,

$$\begin{vmatrix}\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\end{vmatrix}=0$$

which on simplifying reduces to :

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \quad ...(6)$$

If $f(u, v)$ is a solution of the equation (6), differentiating $u(x, y, z) = A$ and $v(x, y, z) = B$

we have, $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0$ ... (7)

and $\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0$ ... (8)

Eliminating dx, dy, dz from (7) and (8) we have,

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \quad ...(9)$$

Comparing (5) and (9), we may write,

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P}=\frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q}=\frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R}=k \text{ [say]} \quad ...(10)$$

From (10) we have,

$$\left.\begin{aligned}\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}&=kP\\ \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}&=kQ\\ \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}&=kR\end{aligned}\right\} \quad ...(11)$$

Using (11) in (6) we have, $kPp + kQq = kR$ *i.e.*, $Pp + Qq = R$

Hence we may conclude that if $u \equiv u(x, y, z) = A$ and $v \equiv v(x, y, z) = B$ are two independent solutions of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, then $f(u, v) = 0$ [or $u = f(v)$] is also a solution of the equation $Pp + Qq = R$ for A, B being two arbitrary constants and f being an arbitrary function.

Thus any first order partial differential equation of the form $Pp + Qq = R$ can be solved by finding the solutions of the Lagrange's auxiliary equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$. However the solution technique of the differential equations of the form $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ have already been discussed in simultaneous differential equation chapter.

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Illustrative Examples

1. Solve : $p + q = 1$

Solution Here the given differential equation is $p + q = 1$ ... (1)

So the Lagrange's auxiliary equations are $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$ ... (2)

Considering first two, we get, $dx = dy$

Integrating $x = y + A$; where A is an arbitrary constant.

Again considering first and third, we get, $dx = dz$

Integrating $x = z + B$; where B is an arbitrary constant.

So the required general solution is $A = f(B)$ *i.e.*, $x - y = f(x - z)$, where f is an arbitrary function.

2. Solve : $p\tan x + q\tan y = \tan z$

Solution Here the given partial differential equation is

$p\tan x + q\tan y = \tan z$ ... (1)

The differential equation (1) is of the form $Pp + Qq = R$ and so it can be solved by Lagrange's method. Therefore, the Lagrange's auxiliary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad ... (2)$$

Considering first two fractions, we get,

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow a + x\,dx = a + y\,dy$$

Integrating we get,

$$\log\sin x = \log\sin y + \log A \Rightarrow \log\frac{\sin x}{\sin y} = \log A \Rightarrow \frac{\sin x}{\sin y} = A$$

Again considering first and third fraction we get,

$$\frac{dx}{\tan x} = \frac{dz}{\tan z} \Rightarrow a + x\,dx = a + z\,dz \Rightarrow \log\sin x = \log\sin z + \log B$$

$$\Rightarrow \frac{\sin x}{\sin z} = \log B \Rightarrow \frac{\sin x}{\sin z} = B$$

So, the general solution is $A = f(B)$ *i.e.*, $\frac{\sin x}{\sin y} = f\left(\frac{\sin x}{\sin z}\right)$, where f is an arbitrary function.

3. Solve : $ayzp + bzxq = cxy$.

Solution Here the given partial differential equation is $ayzp + bzxq = cxy$ ... (1)

The differential equation (1) is of the form $Pp + Qq = R$ and so it can be solved by Lagrange's method. Therefore the Lagrange's auxiliary equation are

$$\frac{dx}{ayz} = \frac{dy}{bzx} = \frac{dz}{cxy} \quad ... (2)$$

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Considering the first two fractions, we get,

$$\frac{dx}{ayz} = \frac{dy}{bzx} \Rightarrow aydy = bxdx$$

Integrating we get,

$ay^2 = bx^2 + A$; where A is an arbitrary constant.

Again considering the last two fractions,

$$\frac{dy}{bzx} = \frac{dz}{cxy} \Rightarrow cydy = bzdz$$

Integrating, we get

$cy^2 = bz^2 + B$; where B is an arbitrary constant.

Hence the general solution is

$A = f(B)$ *i.e.*, $(ay^2 - bx^2) = f(cy^2 - bz^2)$, where f is an arbitrary function.

4. **Solve : $pyz + qzx = xy$**

Solution Here the given partial differential equation is $pyz + qzx = xy$ ... (1)

The equation (1) can be solved by Lagrange's method and so the Lagrange's auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \quad \text{... (2)}$$

Considering the first two fractions of the equation (2) we have,

$$\frac{dx}{yz} = \frac{dy}{zx} \quad i.e., \; xdx = ydy \quad \text{... (3)}$$

On integrating (3), we have, $x^2 = y^2 + A$ ... (4)

where A is an arbitrary constant.

Again considering the last two fractions of the equation (2) we have,

$$\frac{dy}{zx} = \frac{dz}{xy} \quad i.e., \frac{dy}{z} = \frac{dz}{y} \quad i.e., \; y\,dy = z\,dz \quad \text{... (5)}$$

On integrating (5) we have, $y^2 = z^2 + B$ ... (6)

B being an arbitrary constant.

Hence the required general solution is $A = f(B)$ *i.e.*, $x^2 - y^2 = f(y^2 - z^2)$, where f is an arbitrary function.

5. **Solve : $p\sec^2 x + q\sec^2 y = \sec^2 z$**

Solution Here the given differential equation is $p\sec^2 x + q\sec^2 y = \sec^2 z$ ... (1)

The Lagrange's auxiliary equations of (1) is

$$\frac{dx}{\sec^2 x} = \frac{dy}{\sec^2 y} = \frac{dz}{\sec^2 z} \quad \text{... (2)}$$

Omitting the third fraction from equation (2) we have,

$$\frac{dx}{\sec^2 x} = \frac{dy}{\sec^2 y} \quad \text{... (3)}$$

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Integrating (3) we have, $\tan z = \tan y + A$ *i.e.*, $A = \tan z - \tan y$

where A is an arbitrary constant.

Again omitting the second fraction from equation (2) we have,

$$\frac{dx}{\sec^2 x} = \frac{dz}{\sec^2 z} \quad \text{... (4)}$$

Integrating (4) we have, $\tan x = \tan z + B$

i.e., $B = \tan x - \tan z$, where B is an arbitrary constant.

Hence the required general solution is

$A = f(B)$ *i.e.*, $\tan z - \tan y = f(\tan x - \tan z)$, where f is an arbitrary function.

6. **Solve : $2p - 3q = z + \tan(3x + 2y)$**

Solution Here the given differential equation is $2p - 3q = z + \tan(3x + 2y)$ ... (1)

Lagrange's auxiliary equations of (1) are

$$\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{z + \tan(3x + 2y)} \quad \text{... (2)}$$

Omitting the third fraction we have from (2),

$$\frac{dx}{2} = \frac{dy}{-3} \quad i.e., \quad 3dx + 2dy = 0 \quad \text{... (3)}$$

Integrating (3) we have, $3x + 2y = A$

where A is an arbitrary constant.

Again omitting the second fraction of (2) we have,

$$\frac{dx}{2} = \frac{dz}{z + \tan(3x + 2y)} \quad i.e., \quad \frac{dx}{2} = \frac{dz}{z + \tan A} \quad \text{[Using (3)]} \quad \text{... (4)}$$

Integrating (4) we get,

$\frac{x}{2} = \log(z + A) + \frac{B}{2}$, where B is an arbitrary constant,

i.e., $x = 2\log(3x + 2y + z) + B$ [From (3)]

i.e., $B = x - 2\log(3x + 2y + z)$

Thus the required general solution is $B = f(A)$

i.e., $x = 2\log(3x + 2y + z) + f(3x + 2y)$, where f is an arbitrary function.

7. **Solve : $\left(\frac{1}{z} - \frac{1}{y}\right)p + \left(\frac{1}{x} - \frac{1}{z}\right)q = \frac{1}{x} - \frac{1}{y}$** [BH 2007]

Solution Here the given differential equation is

$$\left(\frac{1}{z} - \frac{1}{y}\right)p + \left(\frac{1}{x} - \frac{1}{z}\right)q = \frac{1}{x} - \frac{1}{y} \quad \text{... (1)}$$

The Lagrange's auxiliary equations of (1) are

$$\frac{yz\,dx}{y - z} = \frac{zx\,dy}{z - x} = \frac{xy\,dz}{x - y} \quad \text{... (2)}$$

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Clearly we have from (2),

$$\frac{yz\,dx}{y-z}=\frac{zx\,dy}{z-x}=\frac{xy\,dz}{x-y}=\frac{yz\,dx+zx\,dy+xy\,dz}{(y-z)+(z-x)+(x-y)}=\frac{yz\,dx+zx\,dy+xy\,dz}{0}$$

i.e., $yz\,dx+zx\,dy+xy\,dz=0$ *i.e.*, $d(xyz)=0$

Integrating we have, $xyz = A$ ... (3)

A being an arbitrary constant.

Next multiplying numerator and denominator of each fraction of (2) respectively by $\frac{1}{yz}$, $\frac{1}{zx}$ and $\frac{1}{xy}$, we have from (1)

$$\frac{dx}{\frac{y-z}{yz}}=\frac{dy}{\frac{z-x}{zx}}=\frac{dz}{\frac{x-y}{xy}}=\frac{dx+dy+dz}{\frac{y-z}{yz}+\frac{z-x}{zx}+\frac{x-y}{xy}}=\frac{dx+dy+dz}{0}\quad i.e.,\ d(x+y+z)=0$$

Integrating we have, $x+y+z=B$ ... (4)

From (2) and (3) the required general solution is $A=f(B)$ *i.e.*, $xyz=f(x+y+z)$, where f is an arbitrary function.

8. **Solve : $(y+z)p+(z+x)q=x+y$**

Solution Here the given differential equation is $(y+z)p+(z+x)q=x+y$ ... (1)

The Lagrange's auxiliary equations of (1) are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y} \quad \text{... (2)}$$

Each ratio of (2) is equal to

$$\frac{dx-dy}{y+z-z-x}=\frac{dy-dz}{z+x-x-y}\Rightarrow\frac{d(x-y)}{-(x-y)}=\frac{d(y-z)}{-(y-z)} \quad \text{... (3)}$$

Integrating we get,

$\log(x-y)=\log(y-z)+\log A$, where A is an arbitrary constant.

i.e., $x-y=A(y-z)$ giving $A=\frac{x-y}{y-z}$

Now from (2), again we have, each ratio is equal to :

$$\frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)}=\frac{d(x+y+z)}{2(x+y+z)} \quad \text{... (4)}$$

Now from (3) and (4), we get

$\log(x-y)+\frac{1}{2}\log(x+y+z)=\log B$, where B is an arbitrary constant.

i.e., $(x-y)\sqrt{x+y+z}=B$

Hence the general solution is $A=f(B)$ *i.e.*, $(x-y)=(y-z)\,f((x-y)\sqrt{x+y+z}\,)$

$(x-y)=A(y-z)$, $(x-y)\sqrt{x+y+z}=B$, where f is an arbitrary function.

9. **Solve : $pz(x+y)+qz(x-y)=x^2+y^2$**

Solution Here the given differential equation is $pz(x+y)+qz(x-y)=x^2+y^2$ ... (1)

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The Lagrange's auxiliary equations are

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} \quad \text{... (2)}$$

Multiplying the numerators and denominators of the three fractions of (2) by $x, -y, -z$ respectively we have,

$$\frac{x\,dx}{xz(x+y)} = \frac{-y\,dy}{-yz(x-y)} = \frac{-z\,dz}{-z(x^2+y^2)} = \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

i.e., $x\,dx - y\,dy - z\,dz = 0$

Integrating we get, $x^2 - y^2 - z^2 = A$, where A is an arbitrary constant. ...(3)

Next multiplying the numerators and denominators of the three fractions of (2) by y, x and $-z$ respectively we have,

$$\frac{y\,dx}{yz(x+y)} = \frac{x\,dx}{xz(x-y)} = \frac{-z\,dz}{-z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}$$

i.e., $y\,dx + x\,dy - z\,dz = 0$ i.e., $d(xy) - z\,dz = 0$

Integrating we get, $xy - \frac{z^2}{2} = \frac{B}{2}$ ...(4)

i.e., $2xy - z^2 = B$, where B is an arbitrary constant.

Combining (3) and (4), the general solution of the given differential equation can be written as

$A = f(B)$ i.e., $x^2 - y^2 - z^2 = f(2xy - z^2)$, where f is an arbitrary function.

10. **Solve : $p(z^2 - 2yz - y^2) + q(y + z) = y - z$**

Solution Here the given differential equation is

$$p(z^2 - 2yz - y^2) + q(y + z) = y - z \quad \text{... (1)}$$

The Lagrange's auxiliary equations are

$$\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z} \quad \text{... (2)}$$

Multiplying the numerator and denominator of 2nd and 3rd fraction of (2) by y and z respectively, we have

$$\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{y\,dy}{y(y+z)} = \frac{z\,dz}{z(y-z)} = \frac{x\,dx + y\,dy + z\,dz}{z^2 + 2yz - y^2 + y(y+z) + z(y-z)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e., $x\,dx + y\,dy + z\,dz = 0$

Integrating we have, $x^2 + y^2 + z^2 = A$, where A is an arbitrary constant. ... (3)

Again from 2nd and 3rd fraction of (2) we have,

$\frac{dy}{y+z} = \frac{dz}{y-z}$ i.e., $(y + z)dz - (y - z)dy = 0$

i.e., $(y\,dz + z\,dy) + z\,dz - y\,dy = 0$ i.e., $d(yz) - y\,dy + z\,dz = 0$

Integrating we have, $2yz - y^2 + z^2 = B$, where B is an arbitrary constant. ... (4)

Hence from (3) and (4), the general solution of the given differential equation (1) is

$A = f(B)$ i.e., $x^2 + y^2 + z^2 = f(2yz - y^2 + z^2)$, where f is an arbitrary function.

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11. Solve : $px(y^2 - z^2) + qy(z^2 - x^2) = z(x^2 - y^2)$

Solution Here the given differential equation is

$$px(y^2 - z^2) + qy(z^2 - x^2) = z(x^2 - y^2) \quad \text{... (1)}$$

The Lagrange's auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \quad \text{... (2)}$$

From (2) we have

$$\frac{\frac{dx}{x}}{y^2 - z^2} = \frac{\frac{dy}{y}}{z^2 - x^2} = \frac{\frac{dz}{z}}{x^2 - y^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

i.e., $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

Integrating we get, $\log x + \log y + \log z = \log A$

i.e., $\log xyz = \log A$ giving $xyz = A$, where A is an arbitrary constant. ...(3)

Again from (2) we have,

$$\frac{x\,dx}{x^2(y^2 - z^2)} = \frac{y\,dy}{y^2(z^2 - x^2)} = \frac{z\,dz}{z^2(x^2 - y^2)} = \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e., $x\,dx + y\,dy + z\,dz = 0$

Integrating we get, $x^2 + y^2 + z^2 = B$, where B is an arbitrary constants. ...(3)

Hence the required general solution of the given differential equation is $A = f(B)$

i.e., $xyz = f(x^2 + y^2 + z^2)$ where f is an arbitrary function.

12. Solve : $px(2y^4 - z^4) + qy(z^4 - 2x^4) = z(x^4 - y^4)$.

Solution Here the given differential equation is

$$px(2y^4 - z^4) + qy(z^4 - 2x^4) = z(x^4 - y^4) \quad \text{... (1)}$$

The Lagrange's auxiliary equations are

$$\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)} \quad \text{... (2)}$$

From (2) we have,

$$\frac{x^3dx}{x^4(2y^4 - z^4)} = \frac{y^3dy}{y^4(z^4 - 2x^4)} = \frac{z^3dz}{z^4(x^4 - y^4)}$$

$$= \frac{x^3dx + y^3dy + z^3dz}{x^4(2y^4 - z^4) + y^4(z^4 - 2x^4) + z^4(x^4 - y^4)}$$

$$= \frac{x^3dx + y^3dy + z^3dz}{0}$$

i.e., $x^3dx + y^3dy + z^3dz = 0$

Integrating we have, $x^4 + y^4 + z^4 = A$, where A is an arbitrary constant. ... (3)

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Again from (2) we have,

$$\frac{yz^2dx}{xyz^2(2y^4-z^4)} = \frac{xz^2dy}{xyz^2(z^4-2x^4)} = \frac{2xyz\,dz}{2xyz^2(x^4-y^4)} = \frac{yz^2dx+xz^2dy+2xyz\,dz}{xyz^2(2y^4-z^4+z^4-2x^4+2x^4-2y^4)}$$

$$= \frac{yz^2dx+xz^2dy+2xyz\,dz}{0}$$

i.e., $yz^2dx + xz^2dy + 2xyz\,dz = 0$ i.e., $d(xyz^2) = 0$

Integrating we get, $xyz^2 = B$, where B is an arbitrary constant. ... (4)

Hence combining (3) and (4) required general solution are given by $A = f(B)$

i.e., $x^4 + y^4 + z^4 = f(xyz^2)$, where f is an arbitrary function.

13. Solve : $p(x^2 - y^2 - z^2) + 2qxy = 2xz$

Solution Here the given differential equation is $p(x^2 - y^2 - z^2) + 2qxy = 2xz$ ... (1)

The Lagrange's auxiliary equations are $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ ...(2)

From the second and third factor of (2) we have,

$$\frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{i.e.,} \quad \frac{dy}{y} = \frac{dz}{z}$$

Integrating we get $\log y = \log Az$, where A is an arbitrary constant.

i.e., $y = Az$ i.e., $A = y/z$ ... (3)

Again multiplying numerator and denominator of each fraction of (2) respectively by x, y and z we have,

$$\frac{x\,dx}{x(x^2-y^2-z^2)} = \frac{y\,dy}{2xy^2} = \frac{z\,dz}{2xz^2} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2-y^2-z^2)+2xy^2+2xz^2}$$

i.e., $\frac{y\,dy}{2xy^2} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$ [considering 2nd and 4th fraction]

i.e., $\frac{dy}{2y} = \frac{d(x^2+y^2+z^2)}{2(x^2+y^2+z^2)}$ i.e., $\frac{dy}{y} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$

Integrating we get,

$\log(x^2 + y^2 + z^2) = \log By$, where B is an arbitrary constant.

i.e., $x^2 + y^2 + z^2 = By$ i.e., $B = \frac{(x^2+y^2+z^2)}{y}$ ... (4)

From (3) and (4), the required general solution of the differential equation (1) can be written as $A = f(B)$ i.e., $y = zf\left(\frac{x^2+y^2+z^2}{y}\right)$, where f is an arbitrary function.

14. Solve : $p(x^2 + y^2) + 2qxy = (x + y)z$

Solution Here the given differential equation is $p(x^2 + y^2) + 2qxy = (x + y)z$ ... (1)

The Lagrange's auxiliary equations are $\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z}$ ...(2)

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Considering first two fractions of the equation (2), we have,

$$\frac{dx}{x^2+y^2}=\frac{dy}{2xy} \quad i.e., (x^2+y^2)dy = 2xy\,dx$$

$$i.e.,\ y^2dy + (x^2dy - 2xy\,dx) = 0 \quad i.e.,\ dy = \frac{2xy\,dx - x^2dy}{y^2} \quad i.e.,\ dy = d\left(\frac{x^2}{y}\right)$$

Integrating we get, $y + A = \frac{x^2}{y}$, A being an arbitrary constant.

i.e., $x^2 = y^2 + Ay$ *i.e.*, $A = (x^2 - y^2)/y$ ... (3)

Again multiplying the numerator and denominator of each fraction of (1) respectively by z, z and $(x+y)$ we have,

$$\frac{z\,dx}{z(x^2+y^2)}=\frac{z\,dy}{z\cdot 2xy}=\frac{(x+y)dz}{z(x^2+y^2)}=\frac{z\,dx+z\,dy-(x+y)dz}{z(x^2+y^2)+z\cdot 2xy-(x+y)^2\cdot z}=\frac{z\,dx+z\,dy-(x+y)dz}{0}$$

$$i.e.,\ z(dx+dy)-(x+y)dz=0 \quad i.e.,\ \frac{d(x+y)}{x+y}=\frac{dz}{z}$$

Integrating we get,

$\log(x+y) = \log(Bz)$, where B is an arbitrary constant.

i.e., $x + y = Bz$ *i.e.*, $B = \frac{x+y}{z}$ ... (4)

Combining (3) and (4), the general solution of the given differential equation is

$A = f(B)$ *i.e.*, $x^2 - y^2 = yf\left(\frac{x+y}{z}\right)$, where f is an arbitrary function.

15. **Solve : $(bz - cy)p + (cx - az)q = ay - bz$**

Solution Here the given equation is $(bz - cy)p + (cx - az)q = ay - bz$ ... (1)

The Lagrange's auxiliary equations are $\frac{dx}{bz-cy}=\frac{dy}{cx-az}=\frac{dz}{ay-bz}$ ... (2)

Multiplying numerator and denominator of each fraction of (2) by x, y and z respectively we have,

$$\frac{x\,dx}{x(bz-cy)}=\frac{y\,dy}{y(cx-az)}=\frac{z\,dz}{z(ay-bx)}=\frac{x\,dx+y\,dy+z\,dz}{x(bz-cy)+y(cx-az)+z(ay-bx)}=\frac{x\,dx+y\,dy+z\,dz}{0}$$

i.e., $x\,dx + y\,dy + z\,dz = 0$

Integrating we get, $x^2 + y^2 + z^2 = A$, where A is an arbitrary constant. ... (3)

Again multiplying numerator and denominator of each fraction of (2) by a, b and c respectively we have,

$$\frac{a\,dx}{a\,bz-a\,cy}=\frac{b\,by}{b\,cx-a\,bz}=\frac{c\,dz}{a\,cy-b\,cx}=\frac{a\,dx+b\,dy+c\,dz}{a\,bz-a\,cy+b\,cx-a\,bz-a\,cy-b\,cx}=\frac{a\,dx+b\,dy+c\,dz}{0}$$

i.e., $a\,dx + b\,dy + c\,dz = 0$, *i.e.*, $d(ax + by + dz) = 0$

Integrating we have, $ax + by + cz = B$, where B is an arbitrary constant. ... (4)

Combining (3) and (4), required general solution of the given differential equation is

$A = f(B)$ *i.e.*, $x^2 + y^2 + z^2 = f(ax + by + cz)$, f is an arbitrary function.

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16. **Solve : $p(y+z)+q(z+x)=x+y$**

Solution Here the given differential equation is $p(y+z)+q(z+x)=x+y$ ... (1)

The Lagrange's auxiliary equations are $\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$ ... (2)

From (2) we have,

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}=\frac{dx-dy}{(y+z)-(z+x)}=\frac{dy-dz}{(z+x)-(x+y)}=\frac{dx+dy+dz}{2(x+y+z)} \quad ...(3)$$

i.e., $\frac{d(x-y)}{-(x-y)}=\frac{d(y-z)}{-(y-z)}$ [Considering 4th and 5th fraction]

i.e., $\frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}$

Integrating we have, $\log(x-y)=\log A+\log(y-z)$

where A is an arbitrary constant.

i.e., $\log(x-y)=\log A(y-z)$ ... (4)

i.e., $x-y=A(y-z)$ i.e., $A=\frac{x-y}{y-z}$

Again considering 5th and 6th fraction of (3), we have,

$$\frac{d(y-z)}{-(y-z)}=\frac{d(x+y+z)}{2(x+y+z)} \quad i.e., \quad \frac{2d(y-z)}{y-z}+\frac{d(x+y+z)}{x+y+z}=0$$

Integrating we have, $2\log(y-z)+\log(x+y+z)=\log B$

where B is an arbitrary constant. ... (5)

i.e., $\log(y-z)^2(x+y+z)=\log B$ i.e., $(y-z)^2(x+y+z)=B$

Combining (4) and (5), the general solution of the given differential equation is

$A=f(B)$ i.e., $x-y=(y-z)\,f((y-z)^2(x+y+z))$, where f is an arbitrary function.

17. **Solve : $p\sin(x+y)+q\cos(x+y)=z$**

Solution Here the given differential equation is $p\sin(x+y)+q\cos(x+y)=z$ ... (1)

The Lagrange's auxiliary equations are $\frac{dx}{\sin(x+y)}=\frac{dy}{\cos(x+y)}=\frac{dz}{z}$ ... (2)

From the 1st and 2nd fraction of (2) we have, ... (3)

$$\frac{dx}{\sin(x+y)}=\frac{dy}{\cos(x+y)} \quad i.e., \quad \frac{dy}{dx}=\cot(x+y)$$

To solve (3) we substitute, $v=x+y$. Then we have,

$$\frac{dv}{dx}-1=\cot v \quad i.e. \quad \frac{dv}{1+\cot v}=dx$$

$$i.e., \; dx=\frac{\sin v\,dv}{\sin v+\cos v}=\frac{1}{2}\frac{(\sin v-\cos v)(\sin v+\cos v)}{\sin v+\cos v}$$

$$i.e., \; dx=\frac{1}{2}\left[dv-\frac{\cos v-\sin v}{\sin v+\cos v}dv\right] \quad i.e., \quad 2dx=dv-\frac{d(\sin v+\cos v)}{\sin v+\cos v}$$

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Integrating we have,

$2x = v - \log(\sin v + \cos v) - \log A$, where A is an arbitrary constant.

i.e., $\log A(\sin v + \cos v) = v - 2x$

i.e., $A\{\sin(x + y) + \cos(x + y)\} = e^{y-x}$ [since $v = x + y$] ... (4)

Again from (1) we have,

$$\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z} = \frac{dx+dy}{\sin(x+y)+\cos(x+y)} \quad \text{... (5)}$$

Considering 3rd and 4th fraction of (4), we have,

$$\frac{dz}{z} = \frac{d(x+y)}{\sin(x+y)+\cos(x+y)} = \frac{dv}{\sin v + \cos v} \quad \text{[since } v = x + y\text{]}$$

$$= \frac{dv}{\sqrt{2}(\sin v \cos\frac{\pi}{4} + \cos v \sin\frac{\pi}{4})} = \frac{dv}{\sqrt{2}\sin(v+\frac{\pi}{4})}$$

i.e., $\frac{\sqrt{2}\,dz}{z} = \operatorname{cosec}(v+\frac{\pi}{4})dv$

Integrating we get,

$\sqrt{2}\log = \log B - \log\{\operatorname{cosec}(v+\frac{\pi}{4}) + \cot(v+\frac{\pi}{4})\}$, where B is an arbitrary constant.

i.e., $z^{\sqrt{2}}\{\operatorname{cosec}(x+y+\frac{\pi}{4})\cot(x+y+\frac{\pi}{4})\} = B$ [since $v = x + y$] ...(6)

Hence the required general solution of the given differential equation can be written as $A = f(B)$, where A and B are given by (4) and (6) and f is an arbitrary function.

18. **Solve : $p(x^2 - yz) + q(y^2 - zx) = z^2 - xy$**

Solution Here the given differential equation is $p(x^2 - yz) + q(y^2 - zx) = z^2 - xy$ ... (1)

Lagrange's auxiliary equations are $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$ ...(2)

From (2) we have,

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} = \frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz} \quad \text{... (3)}$$

From 4th and 5th fractions of (3) we have,

$$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy} \text{ i.e., } \frac{d(x-y)}{(x+y+z)(x-y)} = \frac{d(y-z)}{(x+y+z)(y-z)} \text{ i.e., } \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating we get, $\log(x - y) = \log(y - z) + \log(A)$, where A is an arbitrary constant.

i.e., $x - y = A(y - z)$ giving $A = \frac{x-y}{y-z}$ ...(4)

Again from last two fractions of equation (3) we have,

$$\frac{dy-dz}{y^2-zx-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}$$

i.e., $\frac{d(y-z)}{(x+y+z)(y-z)} = \frac{d(z-x)}{(x+y+z)(z-x)}$ *i.e.*, $\frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x}$

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Integrating we get, $\log(y-z) = \log(z-x) + \log B$, where B is an arbitrary constant.

i.e., $y - z = B(z - x)$ giving $B = \dfrac{y-z}{z-x}$ ... (5)

Thus the required general solution of the given differential equation is $A = f(B)$

i.e., $x - y = (y - z)\, f\left(\dfrac{y-z}{z-x}\right)$, where f is an arbitrary function of x, y, z.

---

19. **Solve : $pxy + qy^2 = xyz - 2x^2$**

Solution Here the given differential equation is $pxy + qy^2 = xyz - 2x^2$ ... (1)

The Lagrange's auxiliary equations of (1) is $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{xyz - 2x^2}$ ... (2)

Considering first two fractions of differential equation (2), we have,

$$\frac{dx}{xy} = \frac{dy}{y^2} \quad i.e., \quad \frac{dx}{x} = \frac{dy}{y}$$

Integrating we get $\log x = \log A + \log y$, A being an arbitrary constant. ... (3)

i.e., $x = Ay$ giving $A = x/y$

Again considering 1st and 3rd fraction of the equation (2), we have,

$$\frac{dz}{xyz - 2x^2} \quad i.e., \quad \frac{dx}{1} = \frac{dz}{z - 2\frac{x}{y}} = \frac{dz}{z - 2A} \quad i.e., \quad dx = \frac{dz}{z - 2A} \quad \text{[From (3)]}$$

Integrating we get, $x = \log(z - 2A) + B$, where B is an arbitrary constant.

i.e., $x = \log\left(z - \dfrac{2x}{y}\right) + B$ [From (3)] i.e., $B = x - \log\left(z - \dfrac{2x}{y}\right)$

Hence the general solution of the given differential equation is $A = f(B)$

i.e., $x = yf\left(x - \log\left(z - \dfrac{2x}{y}\right)\right)$, where f is an arbitrary function.

Exercise

- **Solve :**

1. $px - qy = xy$ [BH 2002]
2. $py^2 - xyq = x(z - 2y)$ [KH 2000, NBH 2008]
3. $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$ [KH 2003]
4. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ [CH 2006]
5. $z(x + y)p + z(x - y)q = x^2 + y^2$ [CH 2007]
6. $(y + zx)p - (x + yz)q = x^2 - y^2$ [VH 2007]
7. $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$ [NBH 2007]
8. $(y + z)p + (z + x)q = x + y$ [IAS 1997]
9. $(y^2 + z^2)p - xyq + zx = 0$ [IAS 1990]
10. $z(z^2 + xy)(px - qy) = x^4$ [BH 2008]
11. $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$ [Delhi (H) 2000]
12. $y^2(x - y)p + x^2(y - x)q = z(x^2 + y^2)$ [IAS 1996]

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Answers

1. $xy\log x = f(xy) + z$.
2. $x^2 + y^2 = f(zy - y^2)$
3. $x^2 + y^2 + z^2 = zf\left(\frac{y}{z}\right)$
4. $x - y = (y - z)f(xy + yz + zx)$
5. $x^2 - y^2 - z^2 = f(2xy - z^2)$
6. $x^2 + y^2 - z^2 = f(xy + z)$
7. $x^2 + y^2 - 2z = f(xyz)$
8. $(x - y) = (y - z)f((x - y)^2(x + y + z))$
9. $y = zf(x^2 + y^2 + z^2)$
10. $xy = f(x^4 - 2xyz^2 - z^4)$
11. $xy = zf((x^2 + y^2)z)$
12. $x - y = zf(x^3 + y^3)$

[In each case f is an arbitrary function]

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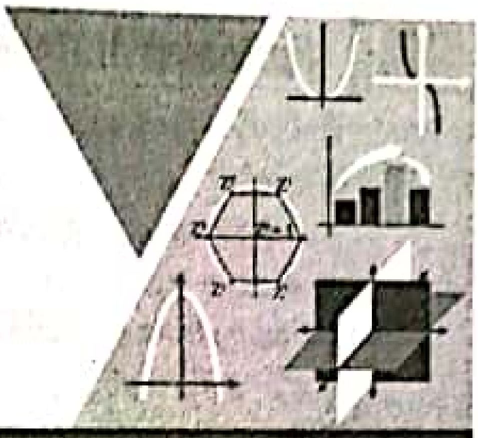
16 CHAPTER

Charpit's Method



Introduction

Charpit's method is a well-known and popular method which can be used to solve a first order non-linear partial differential equation. A non-linear first order partial differential equation should be a function of two independent variable x and y, one dependent variable z and two first order partial derivatives $p\left(=\frac{\partial z}{\partial x}\right)$ and $q\left(=\frac{\partial z}{\partial y}\right)$. Below we describe thoroughly how the Charpit's method is derived and used to solve a non-linear first order partial differential equation.

16.1 Charpit's Method

A first order non-linear partial differential equation should be a functions of x, y, z, p and q and so it can be represented as

$$f(x, y, z, p, q) = 0 \quad ...(1)$$

The basic idea of this method is to select another first order non-linear partial differential equation of the form

$$g(x, y, z, p, q) = a \quad ...(2)$$

(where a is an arbitrary constant) in such a way that from (1) and (2) one can determine the first order partial derivatives

$$p = p(x, y, z, a) \quad ...(3)$$

and $$q = q(x, y, z, a) \quad ...(4)$$

which are to be used in the equation

$$dz = p\,dx + q\,dy \quad ...(5)$$

and after integration we can obtain the solution of the form

$$F(x, y, z, a, b) = 0 \quad ...(6)$$

Where b is an arbitrary constant ariesed due to integration of (5). Therefore (6) is known as the complete integral of the equation (1) which contains two arbitrary constants provided it is assumed that there is an equation of the from (2). We illustrate the method in below :

Differentiating (1) and (2) with respect to x and y partially we have respectively after representing the partial derivatives by suffices.

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$$f_x + pf_z + f_p p_x + f_q q_x = 0 \quad ...(7)$$
$$f_y + qf_z + f_p p_y + f_q q_y = 0 \quad ...(8)$$
$$g_x + pg_z + g_p p_x + g_q q_x = 0 \quad ...(9)$$
$$g_y + qg_z + g_p p_y + g_q q_y = 0 \quad ...(10)$$

Eliminating p_x from (7) and (9) and then q_y from (8) and (10) we respectively have,

$$(f_x g_p - g_x f_p) + p(f_z g_p - g_z f_p) + q_x(f_q g_p - g_q f_p) = 0 \quad ...(11)$$

and $$(f_y g_q - g_y f_q) + q(f_z g_q - g_z f_q) + p_y(f_p g_q - g_p f_q) = 0 \quad ...(12)$$

Again since we know that $q_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = p_y$, then adding the equations (11) and (12) and using $q_x = p_y$ we have, after simplification,

$$f_p g_x + f_q g_y + (pf_p + qf_q)g_z - (f_x + pf_z)g_p - (f_y + qf_z)g_q = 0 \quad ...(13)$$

Equation (13) is a linear first order partial differential equation of Lagrange's type and therefore the auxiliary equations can be obtained by Lagrange's method which are as follows:

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z} = \frac{dg}{0} \quad ...(14)$$

Since for a given problem, g is not a known function, therefore to determine p and q we use only first five equations of (14) and which are given by :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z} \quad ...(15)$$

Equations (15) are known as Charpit's auxiliary equation and can be used on solve a non-linear partial differential equations.

16.2 Steps to solve by Charpit's Method

1. Write down the given non-linear first order differential equation as $f(x, y,, z, p, q) = 0$.
2. Determine f_x, f_y, f_z, f_p and f_q.
3. Put the above partial derivatives in Charpit's auxiliary equation (15) and try to simplify, if possible.
4. Consider two fractions of (15), one of them should be either fourth or fifth fraction or both of them may be selected to find the value of either p or q after integration (which should contain one arbitrary constant) and use this value in the given equation f to determine both of p and q.
5. Use p and q in the relation $dz = p\,dx + q\,dy$ and then integrate to obtain the complete integral which contains two arbitrary constants, obtained due to integration at each step 4 and step 5 and suppose $F(x, y, z, a, b) = 0$ be the complete integral of $f(x, y, z, p, q) = 0$ where a, b are two arbitrary constant.

16.3 General Integral or General Solution

In the complete integral $F(x, y, z, a, b) = 0$ of the equation $f(x, y, z, p, q) = 0$, if we substitute one arbitrary constant as a function of other,

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... = $\phi(a)$, say, then the complete integral reduces to $F(x, y, z, a, \phi(a)) = 0$. However in ... the general integral or general sulution can be obtained from eliminating the ... constant 'a' from $F(x, y, z, a, \phi(a)) = 0$ and the equation $\frac{\partial F}{\partial a} = 0$.

16.4 Singular Integral or Singular Solution

In the complete integral $F(x, y, z, a, b) = 0$ of the equation $f(x, y, z, p, q) = 0$, if we are able to eliminate a and b from the equations $F(x, y, z, a, b) = 0$, $\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial b} = 0$, then the reduced solution is known as the singular solution. However singular solution for each problem does not exist.

Note

> In this chapter we mainly interested to find complete integral of a given partial differential equation. However general integral (or general solution) and singular integral (or singular solution) would be obtained if it is required to do.

16.5 Solution procedure for Some special types of non-linear equations

Some special types of non-linear partial differential equations are as follows :

(i) $f(p, q) = 0$ (ii) $f(p, q, z) = 0$

(iii) $z = px + qy + f(p, q)$ (iv) $f_1(x, p) = f_2(y, q)$

If the given partial differential equation is either of above mentioned four types, then their solution procedures are discussed as below :

- **Form-1 :** Let the given partial differential equation is of the form $f(p, q) = 0$.
- If the from of the given differential equation is $f(p, q) = 0$, then we have, $f_x = f_y = f_z = 0$.

So from Charpit's auxiliary equation (15), we have, ...(16)

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{0} = -\frac{dq}{0}$$

From (16) we have, $dp = 0$ and $dq = 0$

On integration we have, $p = a$ and $q = b$, where a and b are two arbitrary constants.

Now from the given differential equation $f(p, q) = 0$ we have, $f(a, b) = 0$ which can be reduced as $b = f_1(a)$.

Thus we have, $dz = p\,dx + q\,dy = a\,dx + f_1(a)\,dy$.

Integrating we have, $z = ax + f_1(a)\,y + c$, which is the required complete integral for a and c being two arbitrary constants.

- **Illustration-1 :** Solve $ap + bq = c$, where $a, b, c \in \mathbb{R}$

To solve the given differential equation in Charpit's method, we write it as ...(17)

$$f(x, y, z, p, q) = ap + bq - c = 0$$

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Since (17) is independent of x, y, z so the Charpit's auxiliary equation reduces to

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=-\frac{dp}{0}=-\frac{dq}{0} \quad \text{...(18)}$$

From (18) we have, $dp = 0$ and $dq = 0$

On integration we have, $p = c_1$ and $q = c_2$ where c_1 and c_2 are two arbitrary constants.

From (17) we have,

$$ac_1 + bc_2 - c = 0 \quad i.e., \; c_2 = \frac{c-ac_1}{b} \text{ giving } q = \frac{c-ac_1}{b}$$

Again we know that $dz = p\,dx + q\,dy = c_1 dx + \frac{c-ac_1}{b}dy$

Integrating we have, $z = c_1 x + \left(\frac{c-ac_1}{b}\right)y + c_3$, which is the required complete integral where c_1 and c_3 are two arbitrary constants.

- **Form-2 :** Let the given partial differential equation is of the from $f(p, q, z) = 0$.
- If a partial differential equation is of the form $f(p, q, z) = 0$, then we have $f_x = f_y = 0$.

Thus from (15), the auxiliary equations for Charpit's method be becomes as :

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=-\frac{dp}{pf_z}=-\frac{dq}{qf_z} \quad \text{...(19)}$$

From the last two fractions of equation (14), we have,

$\frac{dp}{pf_z}=\frac{dq}{qf_z}$ giving $\frac{dp}{p}=\frac{dq}{q}$ which on integration gives $p = aq$, where a is an arbitrary constants.

Now solving for p and q from the given equation $f(p, q, z) = 0$ and $p = aq$, we may obtain $p = p(z)$ and $q = q(z)$ [containing one arbitrary constant a] so that from the relation $dz = p\,dx + q\,dy = p(z)dx + q(z)dy$ we can easily obtain complete integral, after integration.

Illustration-2 : Solve : $z^2 = p + q$.

To solve the given problem in Charpit's method, we write it as

$$f(x, y, z, p, q) = z^2 - p - q = 0 \quad \text{...(20)}$$

Since f is independent of x and y so $f_x = f_y = 0$. Then the Charpit's auxiliary equation (15) reduces to

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=-\frac{dp}{pf_z}=-\frac{dq}{qf_z}, \text{ since } f_x = f_y = 0 \quad \text{...(21)}$$

From last two fractions of (21), we have, $\frac{dp}{pf_z}=\frac{dq}{qf_z}$

i.e., $\frac{dp}{p}=\frac{dq}{q}$ and on integrating we have, $p = aq$, where a is an arbitrary constant.

Using $p = aq$ in (20) we have, $z^2 - (a + 1)q = 0$ giving, $q = \frac{z^2}{a+1}$ and so $p = \frac{az^2}{a+1}$. Now from $dz = p\,dx + q\,dy$, we have,

$$dz = \frac{az^2}{a+1}dx + \frac{z^2}{a+1}dy \quad i.e., \; \frac{a+1}{z^2}dz = a\,dx + dy$$

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Integrating we have, $ax + y + \dfrac{a+1}{z} = b$, where b is an arbitrary constant. This represents required complete integral of the given problem.

- **Form-3 :** Let the given differential equation be in Clairant's form, i.e., $z = px + qy + f(p, q)$
- To solve by Charpit's method, we write the above equation as

$$f(z, y, x, p, q) = z - px - qy - f(p, q) = 0 \qquad \text{...(22)}$$

From (22) we have, $f_z = 1, f_x = -p, f_y = -q$. Then we have, $f_x + pf_z = f_y + qf_z = 0$.

Thus from (15), the Charpit's auxiliary equations reduces to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0} \qquad \text{...(23)}$$

From last two fractions of (23), we have, $dp = 0$ and $dq = 0$.

On integration we have, $p = a$ and $q = b$, where a and b are two arbitrary constants. Substituting these values in (22) we have,

$z = ax + by + f(a, b)$, which is the required complete integral for a and b being two arbitrary constants. It should further noted that here $dz = p\,dx + q\,dy$, is not required to find the complete integral. However for this form we may easily obtain a singular solution by eliminating a and b from the relations.

$$z = ax + by + f(a, b),\ \frac{\partial z}{\partial a} = 0 \text{ and } \frac{\partial z}{\partial b} = 0.$$

- **Illustration-3 : Find the complete integral of $z = px + qy + \log(pq)$.**

To solve the given partial differential equation by Charpit's method, we write it as follows:

$$f(x, y, z, p, q) \equiv z - px - qy - \log(pd) = 0 \qquad \text{...(24)}$$

From (24) we have, $f_z = 1, f_x = -p, f_y = -q$ so that, $f_x + pf_z = f_y + qf_z = 0$. Hence the Charpit's auxiliary equation (15) reduces to $\dfrac{dx}{f_p} = \dfrac{dy}{f_q} = \dfrac{dz}{pf_p + qf_q} = \dfrac{dp}{0} = \dfrac{dq}{0}$ ...(25)

From last two equations of (25) we have, $dp = 0$ and $dq = 0$, which on integration becomes $p = a$ and $q = b$, where a and b are two arbitrary constants. Substituting these two in (24) we have, $z = ax + by + \log(ab)$, which is the required complete integral for a and b being two arbitrary constants.

- **Form-4 :** The given partial differential equation is separable *i.e.*, if the given problem can be re-written and $f_1(x, p) = f_2(y, q)$.
- In this case, to solve the given problem by Charpit's method, we write it as follows :

$$f(x, y, z, p, q) = f_1(x, p) - f_2(y, q) \qquad \text{...(26)}$$

Thus we have, $f_x = f_{1_x}, f_p = f_{1_p}, f_z = 0, f_y = -f_{2_y}, f_q = -f_{2_q}$

From (15), the Charpit's auxiliary equations becomes as :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z} \text{ i.e., } \frac{dx}{f_{1_p}} = \frac{dy}{f_{2_q}} = \frac{dz}{pf_{1_p} + qf_{2_q}} = -\frac{dp}{f_{1_x}} = -\frac{dq}{f_{2_y}} \qquad \text{...(27)}$$

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Considering first and fourth fractions of (27) we have,

$$\frac{dx}{f_{1_p}} = -\frac{dp}{f_{1_x}} \quad i.e., \ \frac{\partial f_1}{\partial x} dp + \frac{\partial f_1}{\partial p} dx = 0 \quad i.e., \ df_1 = 0$$

Integrating we have, $f_1 = f_1(x, p) = a$, a being an arbitrary constant.

From (26) we have, $f_2(y, q) = a$.

Thus from $f_1(x, p) = a$ and $f_2(y, q) = a$ we can easily determine p and q which are respectively functions x and y. Let $p = f_3(x, a)$ and $q = f_4(y, a)$. So from $dz = p\,dx + q\,dy$, we have, $dz = f_3(x, a)dx + f_4(y, a)dy$, which after integration would give complete integral of given problem.

Note

It should be observed that, same result can be found if we consider second and last fraction of equation (27).

Illustration-4 : Find a complete integral of $x(1 + y^2)p = y(1 + x^2)q$.

The given equation can be re-written as $\frac{xp}{1+x^2} = \frac{yq}{1+y^2}$.

Clearly the equation follows the form $f_1(x, p) = f_2(y, q)$.

To solve the equation by Charpit's method, we write it as follows :

$$f(x, y, z, p, q) = f_1(x, p) - f_2(y, d) = \frac{xp}{1+x^2} - \frac{yq}{1+y^2} = 0 \qquad ...(28)$$

where $f_1(x, p) = \frac{xp}{1+x^2}$ and $f_2(y, q) = \frac{yq}{1+y^2}$

Clearly we have $f_x = f_{1_x}, f_y = -f_{2_y}, f_p = f_{1_p}, f_q = -f_{2_q}$ and $f_z = 0$

Thus the Charpit's auxiliary equation (15) reduces to :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{pf_z + f_x} = -\frac{dq}{qf_z + f_y} \quad i.e., \ \frac{dx}{f_{1p}} = \frac{dy}{-f_{2q}} = \frac{dz}{pf_{1p} - qf_{2q}} = -\frac{dp}{f_{1x}} = \frac{dq}{f_{2y}} \ ... (29)$$

From first and fourth fraction of (29) gives,

$$\frac{dx}{f_{1_p}} + \frac{dp}{f_{1_x}} = 0 \quad i.e., \ \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial p} dp = 0 \text{ giving } df_1 = 0.$$

Integrating we have, $f_1(x, p) = a$, an arbitrary constant.

From (28), we have, $f_1(x, p) = f_2(y, q) = a$

$$i.e., \ \frac{xp}{1+x^2} = a = \frac{yq}{1+y^2} \text{ giving } p = \frac{a(1+x^2)}{x} \text{ and } q = \frac{a(1+y^2)}{y}$$

Again we know that $dz = pdx + qdy$

$$i.e., \ dz = \frac{a(1+x^2)}{x} dx + \frac{a(1+y^2)}{y} dy \quad i.e., \ \frac{dz}{a} - x\,dx - y\,dy = \frac{dx}{x} + \frac{dy}{y}$$

Integrating we have, $\frac{z}{a} - \frac{x^2+y^2}{2} = \log(bxy)$, which is the required complete integral for a and b being two arbitrary constants.

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Illustrative Examples

1. Find a complte integral of $z = px + qy + pq$. **[Mysore 2004]**

Solution First we write the given equation as

$$f(x, y, z, p, q) = z - px - qy - pq = 0 \quad \dots (1)$$

So $f_x = -p, f_y = -q, f_z = 1, f_p = -x - q, f_q = -y - p$.

The auxiliary equation for Charpit's method are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}$$

$$\Rightarrow \frac{dx}{-x-q} = \frac{dy}{-y-p} = \frac{dz}{p(-x-q)+q(-y-p)} = \frac{dp}{p-p} = \frac{dq}{q-q} \quad \dots (3)$$

Taking 4th ratio $dp = 0 \Rightarrow p = c_1$ where c_1 is an arbitrary constant.

Again for 5th ratio $dq = 0 \Rightarrow q = c_2$ where c_2 is an arbitrary constants.

Now putting $p = c_1$ and $q = c_2$ in (1), we get the required congulate intergral

$z = c_1x + c_2y + c_1c_2$, where c_1 and c_2 being arbitrary constants.

2. Find the complete integral of $z = px + qy + p^2 + q^2$ **[IAS 1996]**

Solution To solve the given equation by Charpit's method, it can be written as :

$$f(x, y, z, p, q) \equiv z - px - qy - p^2 - q^2 = 0 \quad \dots (1)$$

We have from (1), $f_x = -p, f_y = -q, f_z = 1, f_p = -x - 2p, f_q = -y - 2q$

Therefore the Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z} \text{ i.e., } \frac{dx}{-p} = \frac{dy}{-q} = \frac{dz}{pf_p + df_q} = -\frac{dp}{-p+p} = -\frac{dq}{-q+q} \quad \dots (2)$$

From (2) we have, $dp = 0$ and $dq = 0$

Integrating we have, $p = c_1, q = c_2$, where c_1 and c_2 are two arbitrary constants.

Using these values in (1) we have, the complete integral as :

$z = c_1x + c_2y + c_1^2 + c_2^2$, where c_1 and c_2 are two arbitrary constants.

3. Find the complete integral of $q = 3p^2$

Solution First we write the given equation as $f(x, y, z, p, q) = 3p^2 - q = 0 \quad \dots (1)$

So, $f_x = 0, f_y = 0, f_z = 0, f_p = 6p, f_q = -1 \quad \dots (2)$

So, the auxiliary equations for Charpit's method are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \dots (3)$$

$$\Rightarrow \frac{dp}{0 + p\cdot 0} = \frac{dq}{0 + q\cdot 0} = \frac{dz}{-6p^2 + q} = \frac{dx}{-6p} = \frac{dy}{1} \text{ [using (1)]}$$

From first ratio, $dp = 0 \Rightarrow p = c_1$ where c_1 is an arbitrary constant.

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Now, from (1) $q = 3c_1^2$ ... (4)

Now, put these values of p and q in $dz = pdx + qdy \Rightarrow dz = c_1 dx + 3c_1^2 dy$

Now integrating we get $z = c_1 x + 3c_1^2 y + c_2$,

which is the required complete integral, when c_1 and c_2 are two arbitrary constants.

4. **Find the complete integral of $z = pq$**

Solution To solve the given partial differential equation by Charpit's method, we write the given differential equation as follows :

$$f(x, y, z, p, q) \equiv z - pq = 0 \quad \text{... (1)}$$

From (1) we have,

$$f_x = 0, f_y = 0, f_z = 1, f_p = -q, f_q = -p$$

Therefore, the auxiliary equations for Charpit's method are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

i.e., $$\frac{dx}{-q} = \frac{dy}{-p} = \frac{dz}{-2pq} = \frac{dp}{-p} = \frac{dq}{-q} \quad \text{... (2)}$$

From first and last equation of (2), we have,

$$\frac{dx}{-q} = \frac{dq}{-q} \text{ giving } dq = dx$$

Integrating we have, $q = c_1 + x$

where c_1 is an arbitrary constant. ...(3)

Again from second and fourth fraction of (2) we have,

$$\frac{dy}{-p} = \frac{dp}{-p} \text{ giving } dp = dy$$

Integrating we have, $p = c_2 + y$

where c_2 is an arbitrary constant. ...(4)

Substituting (3) and (4) in (1), the complete integral can be obtained as

$$z = (c_1 + x)(c_2 + y)$$

where c_1 and c_2 are two arbitrary constants.

5. **Find a complete integral of $pxy + pq + qy = yz$**

Solution We write the given equation as

$$f(x, y, z, p, q) \equiv pxy + pq + qy - yz = 0$$

So, $f_x = py, f_y = q + px - z, f_z = -y, f_p = xy + q, f_q = p + y$ ... (1)

∴ The auxilairy equation for Charpit's method are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}$$

i.e., $$\frac{dx}{xy + q} = \frac{dy}{p + y} = \frac{dz}{pxy + pq + pq + qy} = \frac{dp}{-py + py} = \frac{dq}{-q - px + z + qy}$$

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From 4th function, $dp = 0$ giving $p = c_1$ where c_1 is an arbitrary constant.

Now from (1), $c_1xy + c_1q + qy - yz = 0 \Rightarrow q = \frac{yz - c_1xy}{c_1 + y}$

Now, we know that $dz = p\,dx + q\,dy$ giving $dz = c_1dx + \frac{yz - c_1xy}{c_1 + y}dy$

i.e., $\frac{dz - c_1dx}{z - c_1x} = \frac{y\,dy}{c_1 + y}$ i.e., $\frac{d(z - c_1x)}{z - c_1x} = \frac{(y + c_1) - c_1}{y + c_1}dy$ i.e., $\frac{d(z - c_1x)}{z - c_1x} = dy - \frac{c_1}{c_1 + y}dy$

Now integrating we get,

$\log(z - c_1x) = y - c_1\log(c_1 + y) + \log c_2$, where c_2 is an arbitrary constant.

i.e., $\log(z - c_1x)(c_1 + y)^{c_1} = \log c_2e^y \Rightarrow (z - c_1x)(c_1 + y)^{c_1} = c_2e^y$

which is the required complete integral for c_1 and c_2 being two arbitrary constants.

6. Find the complete integral of the equation $p + q = pq$

Solution To solve the given equation by Charpit's method, it can be written as :

$$f(x, y, z, p, q) = p + q - pq = 0 \quad ...(1)$$

From (1) we have, $f_x = 0, f_y = 0, f_z = 0, f_p = 1 - q, f_q = 1 - p$

The Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

$$\text{i.e., } \frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{p(1-q) + q(1-p)} = -\frac{dp}{0} = -\frac{dq}{0} \quad ...(2)$$

From (2) we have, $dp = 0$

Integrating we have, $p = c_1$, c_1 being an arbitrary constant.

From (1) we have, $q(c_1 - 1) = c_1$ [using $p = c_1$]

i.e., $q = c_1/(c_1 - 1)$

We know that $dz = p\,dx + q\,dy = c_1dx + \frac{c_1}{c_1 - 1}dy$

Integrating we have, the required complete integral as follows :

$z + c_2 = c_1\left(x + \frac{y}{c_1 - 1}\right)$, where c_1 and c_2 are two arbitrary constants.

7. Find the complete integral of the equation $(p + q)(z - px - qy) = 1$

Solution The given partial differential equation we write it as :

$(p + q)(z - px - qy) = 1$ i.e., $z - px - qy = \frac{1}{p+q}$ [assuming $\frac{\partial z}{\partial x} \neq -\frac{\partial z}{\partial y}$]

i.e., $z - px - qy - \frac{1}{p+q} = 0$

Let us consider a function $f(x, y, z, p, q)$ such that,

$$f(x, y, z, p, q) \equiv z - px - qy - \frac{1}{p+q} = 0 \quad ...(1)$$

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Thus from (1) we have, $f_x = -p, f_y = -q, f_z = 1$

The Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

i.e., $$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{0} = -\frac{dq}{0}$$ ... (2)

From (2) we have, $dp = 0$ and $dq = 0$

Integrating we have, $p = c_1$ and $q = c_2$

where c_1 and c_2 are two arbitrary constants.

Thus from (1), required complete integral is

$$z = c_1x + c_2y + \frac{1}{c_1 + c_2} \quad i.e., \ (c_1 + c_2)(z - c_1x - c_2y) = 1.$$

8. Find the complete integral of $py + qx + pq = 0$ [IAS 1990]

Solution To solve the given equation is Charpit's method, we write it as :

$$f(x, y, z, p, q) \equiv py + qx + pq = 0$$ ...(1)

From (1) we have, $f_x = q, f_y = p, f_z = 0, f_p = y + q, f_q = x + p$

The Charpit's auxiliary equations are as below :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

i.e., $$\frac{dx}{y+q} = \frac{dy}{x+p} = \frac{dz}{p(y+q) + q(x+p)} = -\frac{dp}{q} = -\frac{dq}{p}$$ ...(2)

In (2), considering first and fourth fraction we have, $\frac{dx}{y+q} = -\frac{dp}{q}$

i.e., $\frac{dx}{-\frac{qx}{p}} = -\frac{qp}{p}$ [From (1), $p(y + q) = -qx$ and so $y + q = -\frac{qx}{p}$]

i.e., $\frac{dp}{p} = \frac{dx}{x}$

Integrating we have, $\log p = \log x + \log c_1$, c_1 being an arbitrary constant.

i.e., $p = c_1x$.

Substituting $p = c_1x$ in (1) we have, $q = -\frac{py}{x+p} = -\frac{c_1xy}{x+c_1x} = -\frac{c_1}{1+c_1}y$, since $x \neq 0$.

We know that, $dz = p\,dx + q\,dy$ i.e., $dz = c_1x\,dx - \frac{c_1}{1+c_1}y\,dy$

Integrating we have, $z + c_2 = c_1\left\{\frac{x^2}{2} - \frac{y^2}{2(1+c_1)}\right\}$, where c_2 is an arbitrary constant.

i.e., $2(1 + c_1)(z + c_2) = c_1\{(1 + c_1)x^2 - y^2\}$, which is the required complete integral for c_1 and c_2 being two arbitrary constants.

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Instead of considering first and fourth fractions of equation (2), we may consider last two fractions too to find the complete integral, but if the given equation is of the form $f(x, y, p, q) = 0$ or equivalently $\phi_1(x, p) = \phi_2(y, q)$ then if we use first and fourth fractions of Charpit's auxiliary equation then the complete integral can be obtained easily.

Alternative Method

The given equation can be written as :

$$p(q+y) + qx = 0$$

i.e., $\dfrac{p}{x} = -\dfrac{q}{q+y} = c_1$ (say), where c_1 is an arbitrary constant.

i.e., $p = c_1 x$ and $q = -c_1(q+y)$ i.e., $p = c_1 x$ and $q = -\dfrac{c_1 y}{1+c_1}$

We know that $dz = p\,dx + q\,dy$ i.e., $dz = c_1 x\,dx - \dfrac{c_1 y}{1+c_1}\,dy$

Integrating and simplifying we have, $2(1+c_1)(z+c_2) = c_1\{(1+c_1)x^2 - y^2\}$

which in the required complete integral for c_1 and c_2 being two arbitrary constants.

[CH 2003]

9. **Solve : $z^2 = pqxy$ by Charpit's method.**

Solution To solve the given partial differential equation by Charpit's method, we write the given equation as ...(1)

$$f(x, y, z, p, q) = z^2 - pqxy = 0$$

So, $f_x = -pqy$, $f_y = -pqx$, $f_z = 2z$, $f_p = -qxy$, $f_q = -pxy$

The auxiliary equations for Charpit's method are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \qquad ...(2)$$

i.e., $$\frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dz}{2pqxy} = \frac{dx}{qxy} = \frac{dy}{pxy}$$

From (2) we have, $\dfrac{x\,dp + p\,dx}{x(-pqy + 2pz) + pqxy} = \dfrac{y\,dq + q\,dy}{y(-pqx + 2qz) + pqxy}$

i.e., $\dfrac{d(px)}{2pxz} = \dfrac{d(qy)}{2qyz}$ i.e., $\dfrac{d(px)}{px} - \dfrac{d(qy)}{qy} = 0$

Integrating we have, $\log(px) - \log(qy) = \log c_1^2$, where c_1 is an arbitrary constant. ...(3)

i.e., $\dfrac{px}{qy} = c_1^2$ i.e., $p = \dfrac{c_1^2 qy}{x}$

Substituting the value of $p = \dfrac{c_1^2 qy}{x}$ in (1) we have, $z^2 = c_1^2 q^2 y^2$ giving $q = \dfrac{z}{c_1 y}$

Again when $q = \dfrac{z}{c_1 y}$, from (3), $p = \dfrac{c_1 z}{x}$

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We know that $dz = p\,dx + q\,dy = \frac{c_1 z}{x}\,dx + \frac{z}{c_1 y}\,dy$ i.e., $\frac{dz}{z} = \frac{c_1}{x}\,dx + \frac{dy}{y}$,

Integrating we get, $\log z = c_1 \log x + \log y + \log c_2$, where c_2 is an arbitrary constant

i.e., $z = c_2 x^{c_1} y$, which is the required general solution where c_1, c_2 are two arbitrary constants.

10. **Find the complete integral of $x^2p^2 + y^2q^2 = a^2$, where a is a constant.**

Solution To solve the given partial differential equation by Charpit's method, we write the given differential equations as follows :

$$f(x, y, z, p, q) = x^2p^2 + y^2q^2 - a^2 = 0 \quad \text{...(1)}$$

From (1) we have,

$$f_x = 2px^2,\ f_y = 2yq^2,\ f_z = 0,\ f_p = 2x^2p,\ f_q = 2y^2q$$

Now the auxiliary equations for Charpit's method are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

i.e., $$\frac{dx}{2x^2p} = \frac{dy}{2y^2q} = \frac{dz}{2x^2p^2 + 2y^2q^2} = \frac{dp}{-2xp^2} = \frac{dq}{-2yq^2} \quad \text{...(2)}$$

From second and last fraction of (2) we have, $\frac{dy}{2y^2q} = \frac{dq}{-2yq^2}$ i.e., $\frac{dq}{q} + \frac{dy}{y} = 0$

Integrating we have, $\log qy = \log c_1$, where c_1 is an arbitrary constant

i.e., $dy = c_1$ ...(3)

Using $qy = c_1$ is (1) we have, $x^2p^2 = a^2 - c_1^2$ giving $xp = \sqrt{a^2 - c_1^2}$ ...(4)

Again we know that, $dz = p\,dx + q\,dy = \frac{\sqrt{a^2 - c_1^2}}{x}\,dx + \frac{c_1}{y}\,dy$ [From (3) and (4)]

Integrating we have,

$$z = \sqrt{a^2 - c_1^2}\,\log x + c_1 \log y + \log c_2, \text{ where } c_2 \text{ is an arbitrary constants.}$$

i.e., $e^z = c_2 x^{\sqrt{a^2 - c_1^2}} y^{c_1}$, which is the required complete integral, where c_1 and c_2 are two arbitrary constants.

11. **Find the complete integral of $(p + q)(px + qy) = 1$** **[Delhi H 2007]**

Solution To solve the given partial differential equation by Charpit's method, we write the given differential equation as follows :

$$f(x, y, z, p, q) = (p + q)(px + qy) - 1 \quad \text{...(1)}$$

From (1) we have,

$$f_x = (p + q)p,\ f_y = (p + q)q,\ f_z = 0,$$

$$f_p = x(p + q) + px + qy,$$

$$f_q = y(p + q) + px + qy$$

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Therefore, the auxiliary equations for Charpit's method are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

i.e., $$\frac{dx}{x(p+q) + px + qy} = \frac{dy}{y(p+q) + px + qy} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-p(p+q)} = \frac{dq}{-q(p+q)} \quad \text{... (2)}$$

From last two equation of (2), we have

$$\frac{dp}{-p(p+q)} = \frac{dq}{-q(p+q)} \quad \text{i.e.,} \quad \frac{dp}{p} = \frac{dq}{q} \quad \text{...(3)}$$

Integrating we have, $p = c_1 q$, where c_1 is an arbitrary constant

Using (3) in (1) we have, $(1 + c_1)q^2(c_1 x + y) - 1 = 0$ ...(4)

i.e., $q^2 = \frac{1}{(1+c_1)(c_1 x + y)}$ i.e., $q = \frac{1}{\sqrt{(1+c_1)(c_1 x + y)}}$ ...(5)

Using (4) in (3) we have, $p = \frac{c_1}{\sqrt{(1+c_1)(c_1 x + y)}}$

We know that

$$dz = p\,dx + q\,dy = \frac{c_1\,dx}{\sqrt{(1+c_1)(c_1 x + y)}} + \frac{dy}{\sqrt{(1+c_1)(c_1 x + y)}} \quad \text{i.e.,} \quad \sqrt{1+c_1}\,dz = \frac{d(c_1 x + y)}{\sqrt{c_1 x + y}}$$

Integrating we have, $\sqrt{1+c_1}\, z = c_2 + 2\sqrt{c_1 x + y}$ where c_2 is an arbitrary constant.

Therefore the required complete integral is

$\sqrt{1+c_1}\, z = c_2 + 2\sqrt{c_1 x + y}$ where c_1 and c_2 are two arbitrary constants.

[CH 2005]

12. **Find the complete integral of $p^2 x + q^2 y = z$**

Solution To solve the given equation by Charpit's method, we write it as : ...(1)

$$f(x, y, z, p, q) \equiv p^2 x + q^2 y - z = 0$$

From (1), we have, $f_x = p^2, f_y = q^2, f_z = -1, f_p = 2px, f_q = 2qy$

The Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} \quad \text{...(2)}$$

i.e., $$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2p^2 x + 2p^2 y} = \frac{dp}{-(p^2 - p)} = \frac{dq}{-(q^2 - q)}$$

From first and fourth equation of (2) we have,

$$\frac{dx}{2px} = -\frac{dp}{p^2 - p} \quad \text{i.e.,} \quad \frac{dp}{p-1} + \frac{dx}{2x} = 0, \text{ since } p \neq 0$$

Integrating we have, $\log(p-1) + \frac{1}{2}\log x = \log c_1$,

where c_1 is an arbitrary constant. ... (3)

i.e., $(p-1)\sqrt{x} = c_1$ i.e., $p = \frac{c_1}{\sqrt{x}} + 1$

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Again from second and last equation of (2) we have,

$$\frac{dy}{2qy} = -\frac{dq}{q^2 - q} \quad \text{i.e.,} \quad \frac{dq}{q-1} + \frac{dy}{2y} = 0, \text{ since } q \neq 0$$

Integrating we have, $\log(q-1) + \frac{1}{2}\log(y) = \log c_2$,

where c_2 is an arbitrary constant.

$$\text{i.e., } (q-1)\sqrt{y} = c_2 \quad \text{i.e., } q = \frac{c_2}{\sqrt{y}} + 1 \qquad \text{... (4)}$$

Substituting the values from (3) and (4) in (1) we have,

$$z = x\left(\frac{c_1}{\sqrt{x}} + 1\right)^2 + y\left(\frac{c_2}{\sqrt{y}} + 1\right)^2$$

i.e., $z = (c_1 + \sqrt{x})^2 + (c_2 + \sqrt{y})^2$, which is the required complete integral with c_1 and c_2 being two arbitrary constants.

13. **Find the complete integral of $pxy + pq + qy = yz$** **[NBH 2007]**

Solution To solve the given equation by Charpit's method, we write it as :

$$f(x, y, z, p, q) \equiv pxy + pq + qy - yz = 0 \qquad \text{... (1)}$$

From (1) we have,

$$f_x = py, f_y = px + q - z, f_z = -y, f_p = xy + q, f_q = p + y$$

The auxiliary equation for Charpit's method are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\text{i.e., } \frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{p(xy+q) + q(p+y)} = -\frac{dp}{py - py} = -\frac{dq}{px + q - z - qy}$$

$$\text{i.e., } \frac{dx}{q+xy} = \frac{dy}{p+y} = \frac{dz}{pxy + 2pq + qy} = \frac{dp}{0} = -\frac{dq}{qy + z - px - q} \qquad \text{...(2)}$$

From (2) we have, $dp = 0$

Integrating we have, $p = c_1$, c_1 being an arbitrary constant.

Using $p = c_1$ in (1) we have, $c_1xy + (c_1 + y)q - yz = 0$ *i.e.*, $q = \dfrac{y(z - c_1x)}{c_1 + y}$

We know that, $dz = p\,dx + q\,dy = c_1dx + \dfrac{y(z - c_1x)dy}{c_1 + y}$

$$\text{i.e., } \frac{d(z - c_1x)}{z - c_1x} = \frac{y}{c_1 + y}dy = \frac{c_1 + y - c_1}{c_1 + y}dy \quad \text{i.e., } \frac{d(z - c_1x)}{z - c_1x} + c_1\frac{dy}{y + c_1} = dy$$

Integrating we have,

$$\log(z - c_1x) + c_1\log(y + c_1) = y + \log c_2, \; c_2 \text{ being an arbitrary constant.}$$

i.e., $c_2e^y = (z - c_1x)(y + c_1)^{c_1}$, which is the required complete integral with c_1 and c_2 being two arbitrary constants.

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14. Find the complete integral of $q = yzp^2$

Solution To solve the given equation by Charpit's method, it can be written as :

$$f(x, y, z, p, q) = yzp^2 - q = 0 \quad ...(1)$$

We have from (1)

$$f_x = 0, f_y = zp^2, f_z = zp^2, f_p = 2yzp, f_q = -1$$

The auxiliary equations for Charpit's method are :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

$$\text{i.e.,} \quad \frac{dx}{2yzp} = \frac{dy}{-1} = \frac{dz}{2yzp^2 - q} = -\frac{dp}{p^3 y} = -\frac{dq}{zp^2 + qyp^2} \quad ...(2)$$

From second and fourth equation of (2) we have

$$\frac{dy}{-1} = -\frac{dp}{p^3 y} \quad \text{i.e., } p^{-3}dp = y\,dy$$

Integrating we have, $\frac{p^{-2}}{-2} = \frac{y^2}{2} - \frac{c_1}{2}$, c_1 being an arbitrary constant.

$$\text{i.e.,} \quad \frac{1}{p^2} = -(c_1 + y^2) \quad \text{i.e.,} \quad p^2 = \frac{1}{c_1 - y^2} \quad ...(3)$$

$$\text{Using (3) in (1) we have,} \quad q = \frac{yz}{c_1 - y^2} \quad ...(4)$$

We know that $dz = p\,dx + q\,dy = \frac{1}{\sqrt{c_1 - y^2}}dx + \frac{yz}{c_1 - y^2}dy$ [From (3) and (4)]

$$\text{i.e.,} \quad dx = \sqrt{c_1 - y^2}\,dz - \frac{yz}{\sqrt{c_1 - y^2}}dy = d\left(z\sqrt{c_1 - y^2}\right)$$

Integrating we have, $x + c_2 = z\sqrt{c_1 - y^2}$ where c_2 is an arbitrary constant.

Hence the required complete integral is $z\sqrt{c_1 - y^2} = x + c_2$, where c_1 and c_2 are two arbitrary constants.

15. Find the complete integral of $q = px + p^2$

Solution To solve the given equation by Charpit's method, it can be written as :

$$f(x, y, z, p, q) = q - px - p^2 = 0 \quad ...(1)$$

From (1) we have,

$$f_x = -p, f_y = 0, f_z = 0, f_p = -x - 2p, f_q = 1$$

Therefore the Charpit's auxiliary equation are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

$$\text{i.e.,} \quad \frac{dx}{-x - 2p} = \frac{dy}{1} = \frac{dz}{-px - 2p^2 + q} = -\frac{dp}{-p} = \frac{dq}{0} \quad ...(2)$$

From second and fourth equation of (2) we have, $\frac{dy}{1} = \frac{dp}{p}$

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Integrating we have, $\log p = \log c_1 + y$, where c_1 is an arbitrary constant.

i.e., $p = c_1 e^y$ ...(3)

Using (3) in (1) we have, $q = c_1 x e^y + c_1^2 e^{2y}$ ...(4)

We know that,

$$dz = p\,dx + q\,dy = c_1 e^y dx + (c_1 x e^y + c_1^2 e^{2y})dy \quad \text{[From (3) and (4)]}$$

$$= c_1(e^y dx + xe^y dy) + c_1^2 e^{2y} dy = c_1 d(xe^y) + c_1^2 e^{2y} dy$$

Integrating we get, $z = c_2 + c_1 x e^y + \frac{1}{2} c_1^2 e^{2y}$,

which is the required complete integral for c_1 and c_2 being two arbitrary constants.

16. **Find the complete integral, general integral and singular integral of the equation $z = px + qy + pq$**

Solution To find the complete integral by Charpit's method we write the given equation as:

$$f(x, y, z, p, q) \equiv z - px - qy - pq \quad ...(1)$$

From (1) we get, $f_x = -p$, $f_y = -q$, $f_z = 1$, $f_p = -x - q$, $f_q = -y - p$

The Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

i.e., $$\frac{dx}{-x-q} = \frac{dy}{-y-p} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{-p+p} = -\frac{dq}{-q+q}$$

i.e., $$\frac{dx}{x+q} = \frac{dy}{y+p} = -\frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0} \quad ...(2)$$

From (2) we have, $dp = 0$ and $dq = 0$

Integrating we have, $p = c_1$ and $q = c_2$ where c_1 and c_1 are two arbitrary constants.

Substituting those values in (1), we have,

$$z = c_1 x + c_2 y + c_1 c_2 \quad ...(3)$$

which in the required complete integral for c_1 and c_2 are two arbitrary constants.

To find general integral, we substitute $c_2 = g(c_1)$, [where g is an arbitrary function] in the complete integral (3). Then we have from (3),

$$z = c_1 x + g(c_1)(y + c_1) \quad ...(4)$$

Differentiating (4) with respect to c_1 partially we have,

$$0 = x + (y + c_1)g'(c_1) + g(c_1) \quad ...(5)$$

Hence the required general integral (or general solution) of the given differential equation can be obtained by eliminating c_1 from (4) and (5).

To find singular integral, we differentiate partially (3) with respect to c_1 and c_2 successively to obtain,

$$0 = x + c_2 \quad i.e., c_2 = -x \quad ...(6)$$

$$\text{and } 0 = y + c_1 \quad i.e., c_1 = -y \quad ...(7)$$

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Eliminating c_1 and c_2 from (3), (6) and (7) we have, required singular integral (or singular solution) as follows :

$$z = -yx - xy + (-y)(-x) \quad i.e., \; z + xy = 0 \qquad \dots (8)$$

17. Find the complete integral of $p^3 + q^3 - 3pqz = 0$ **[IAS 1991]**

Solution Here the given equation can be written as

$$f(x, y, z, p, q) \equiv p^3 + q^3 - 3pqz \qquad \dots (1)$$

We have from (1), $f_x = 0, f_y = 0, f_z = -3pq, f_p = 3p^2 - 3pz, f_q = 3q^2 - 3pz$

The auxiliary equation for Charpit's method are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

$$i.e., \; \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{3p^2q} = \frac{dq}{3pq^2} \qquad \dots (2)$$

From (2) we have, $\dfrac{dp}{3p^2q} = \dfrac{dq}{3pq^2}$ i.e., $\dfrac{dp}{p} = \dfrac{dq}{q}$

Integrating we have, $p = c_1q$, where c_1 is an arbitrary constant.

From (1) we have, $c_1^3q^3 + q^3 - 3c_1q^2z = 0$

i.e., $(1 + c_1^3)q = 3c_1z$ [assuming $q \neq 0$]

i.e., $q = \dfrac{3c_1z}{1+c_1^3}$ and so $p = \dfrac{3c_1^2z}{1+c_1^3}$

We know that $dz = p\,dx + q\,dy$

$$i.e., \; dz = \frac{3c_1^2z}{1+c_1^3}dx + \frac{3c_1}{1+c_1^3}z\,dy$$

$$i.e., \; (1 + c_1^3)\frac{dz}{z} = 3c_1(c_1dx + dy)$$

Integrating we have,

$(1 + c_1^3)\log z = 3c_1(c_1x + y) + c_2$, which is the required complete integral for c_1 and c_2 being two arbitrary constants.

18. Find the complete integral of the partial differential equation $pz = (p^2 + q^2)x$. **[IAS 2004]**

Solution Here the given equation can be written as follows :

$$f(x, y, z, p, q) = pz - (p^2 + q^2)x = 0 \qquad \dots (1)$$

Therefore from (1) we have,

$$f_x = -(p^2 + q^2), f_y = 0, f_z = p, f_p = z - 2px, f_q = -2qx.$$

Charpit's auxiliary equations are as follows :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

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i.e., $\dfrac{dx}{z-2px} = \dfrac{dy}{-2qz} = \dfrac{dz}{pz-2p^2x-2q^2x} = -\dfrac{dp}{-(p^2+q^2)+p^2} = -\dfrac{dq}{pq}$ ...(2)

From the last two fractions of (2) we have, $-\dfrac{dp}{-q^2} = -\dfrac{dq}{pq}$ i.e., $2p\,dp + 2q\,dq = 0$

Integrating we have, $p^2 + q^2 = c_1^2$, where c_1 is an arbitrary constant. ...(3)

Using (3) in (1) we have, $pz = c_1{}^2x$ i.e., $p = \dfrac{c_1^2x}{z}$ ...(4)

Using (4) in (3) we have, $q_2 = c_1^2 - \dfrac{c_1^4x^2}{z^2}$ i.e., $q = \pm\dfrac{c_1\sqrt{z^2-c_1^2x^2}}{z}$

We know that, $dz = p\,dx + q\,dy = \dfrac{c_1^2x}{z}dx \pm \dfrac{c_1\sqrt{z^2-c_1^2x^2}}{z}dy$

i.e., $\pm\dfrac{zdz - c_1^2xdx}{\sqrt{z^2-c_1^2x^2}} = c_1dy$ i.e., $\pm\dfrac{1}{2}\dfrac{d(z^2-c_1^2x^2)}{\sqrt{z^2-c_1^2x^2}} = c_1dy$

Integrating we have, $\pm\sqrt{z^2-c_1^2x^2} = c_1y + c_2$, c_2 is an arbitrary constant.

i.e., $z^2 - c_1^2x^2 = (c_1y + c_1)^2$ [squaring both sides]

i.e., $z^2 = c_1^2x^2 + (c_1y + c_2)^2$, which is the required complete integral for c_1 and c_2 being two arbitrary constants.

19. **Find a complete integral of $xpq + yq^2 = 1$** [CH 2006]

Solution To solve the given equation by Charpit's method, we write it as :

$f(x, y, z, p, q) \equiv xpq + yq^2 - 1 = 0$ ...(1)

From (1) we have, $f_x = pq, f_y = q^2, f_z = 0, f_p = xq, f_q = xp + 2yq$

The Charpit's auxiliary equations are :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

i.e., $\dfrac{dx}{xq} = \dfrac{dy}{xp+2yq} = \dfrac{dz}{pxq+qxp+2yq^2} = -\dfrac{dp}{pq} = -\dfrac{dq}{q^2}$

i.e., $\dfrac{dx}{xq} = \dfrac{dy}{px+2qy} = \dfrac{dz}{2q(px+qy)} = -\dfrac{dp}{pq} = -\dfrac{dq}{q^2}$ ...(2)

From first and last fractions of (2) we have,

$\dfrac{dx}{xq} = -\dfrac{dq}{q^2}$ i.e., $\dfrac{dx}{x} + \dfrac{dq}{q} = 0$ [assuming $q \neq 0$]

i.e., $q\,dx + x\,dq = 0$ i.e., $d(qx) = 0$

Integrating we have, $qx = c_1$, c_1 being an arbitrary constant.

i.e., $q = \dfrac{c_1}{x}$ ...(3)

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Using (3) in (1) we have, $p = \frac{1}{c_1}\left(1 - y\cdot\frac{c_1^2}{x^2}\right)$ i.e., $p = \frac{1}{c_1} - \frac{c_1 y}{x^2}$ ... (4)

We know that,

$$dz = p\,dx + q\,dy = \left(\frac{1}{c_1} - \frac{c_1 y}{x^2}\right)dx + \frac{c_1}{x}dy \text{ [From (3) and (4)]}$$

$$= \frac{dx}{c_1} + c_1\left(\frac{dy}{x} - \frac{y\,dx}{x^2}\right) = \frac{dx}{c_1} + c_1\, d\left(\frac{y}{x}\right)$$

Integrating we have, $z = \frac{x}{c_1} + \frac{c_1 y}{x} + c_2$, which is required complete integral where c_1 and c_2 are two arbitrary constants.

Exercise

- Find the complete integral for the following partial differential equations by Charpit's method (1-22)

1. $z = px + qy + a^2p + b^2q + 1$
2. $z = px + qy + \sin(pq)$
3. $z - px - qy = \frac{a^2}{pq}$
4. $(p + q)(z - px - py) = 1$
5. $pq = a^2$
6. $p^2 + q^2 = 1$
7. $x(1 + y)p - y(1 + x)q = 0$
8. $p + q = pqz$ [NBH 2007]
9. $qz = (p^2 + q^2)y$ [Delhi (H) 1989]
10. $2x(z^2q^2 + 1) = pz$ [IAS 1998]
11. $2z = p^2 + q^2 + 2(p - x)(q - y)$ [IAS 2002]
12. $z = \sqrt{p} - qy$
13. $pz = (p^2 + q^2)x$ [Delhi (H) 2004]
14. $px + q^2 - q = 0$
15. $z = p^2x + q^2y$ [CH 2005]
16. $pq = x^m y^n z^l$ [IAS 1989, 1994]
17. $z = \frac{p+q}{pq}$
18. $9(p^2z + q^2) = 4$ [IAS 1988]
19. $z(p^2 - q^2) = x - y$ [IAS 1989]
20. $(y - x)(qy - px) = (p - q)^2$ [IAS 1992]
21. $z^2(p^2z^2 + q^2) = 1$ [BH 2010]
22. $z - qy = p(x - q^3)$
23. Find a complete, singular and general integrals of $(p^2 + q^2)y = qz$ [Delhi B.Sc (Prog.) II, 2012, Pune 2010, Kanpur 2014]

Answers

1. $z = c_1x + c_2y + a^2c_1 + b^2c_2 + 1$
2. $z = ax + by + \sin(ab)$
3. $z = c_1x + c_2y + \frac{a^2}{c_1c_2}$
4. $(c_1 + c_2)(z - c_1x - c_2y) = 1$
5. $z = c_1x + \frac{a^2}{c_1}y + c_2$
6. $z = ax + \sqrt{1-a^2}\,y + b$
7. $z = a + b[\log(xy) + x + y]$
8. $z^2 = 2\left(1 + \frac{1}{a}\right)(ax + y) + b$

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9. $z^2 = a^2y^2 + (ax + b)^2$

10. $z^2 = 2x^2(a^2 + 1) + 2ay + b$

11. $2z = (x + a)^2 + (y + b)^2 + 2ab$

12. $yz = ax + 2\sqrt{ay} + b$

13. $z^2 = a^2x^2 + (ay + b)^2$

14. $z = (a - a^2)\log x + ay + b$

15. $\sqrt{1+a}\sqrt{z} = \sqrt{ax} + \sqrt{y} + b$

16. $\dfrac{z^{1-\frac{l}{2}}}{1-\frac{l}{2}} = a \cdot \dfrac{x^{m+1}}{m+1} + \dfrac{y^{n+1}}{a(n+1)} + b$

17. $z^2 = 2(1 + a)\left(x + \dfrac{y}{a}\right) + b$

18. $(z + a^2)^3 = (x + ay + b)^2$

19. $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$

20. $z = a^2(x + y) + axy + b$

21. $(z^2 + a^2)^3 = 9(x + ay + b)^2$

22. $z = ax + by - ab^3$

23. Complete integral : $z^2 - c_1^2y^2 = (c_1x + c_2)^2$ where c_1 & c_2 being arbitrary constant.
Singular integral : $z = 0$
General integral : Eliminating c_1, from $z^2 - c_1^2y^2 = [c_1x + \phi(c_1)]^2$ & $-2c_1y^2 = 2[c_1x + \phi(c_1)]\cdot[x + \phi'(c_1)]$ gives general integral.]

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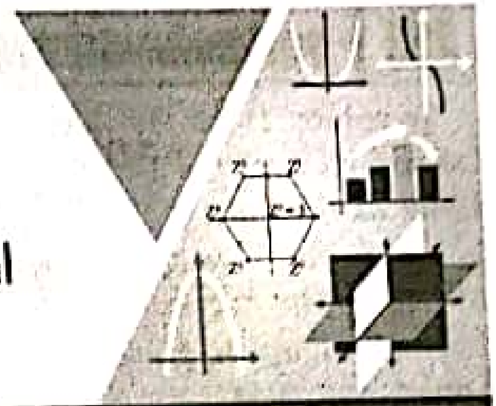
17 CHAPTER

Classification of Second Order Partial Differential Equation



17.1 Reduction of Second Order Differential Equation to its Canonical Form

We know that in any partical differential equation there is at least two independent variables and a dependent variable. Let x and y be two independent variables such that $u \equiv u(x, y)$ is the only dependent variables. Then the general form of a second order partial differential equation can be written as follows :

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u(x, y) = G(x, y) \quad \ldots (1)$$

where A, B, C, D, E, F, G are functions of x and y.

To determine the nature of the partial differential equation (1), we have to determine the sign of the discreminant of (1) which is given by

$$\Delta(x, y) = \{B(x, y)\}^2 - 4A(x, y)C(x, y) \quad \ldots (2)$$

All second order partial differential equations whose general form is given in (1) would be either of following three types depending on the sign of discriminant.

(i) If $\Delta(x, y) < 0$, then the partial differential equation (1) is of elliptic type.

(ii) If $\Delta(x, y) = 0$, then the partial differential equation (1) is of parabolic type and

(iii) If $\Delta(x, y) > 0$, then the partial differential equation (1) is of hyperbolic type.

If we transform the Cartesian (x, y) co-ordinates to another independent system of co-ordinates (ξ, η) where $\xi \equiv \xi(x, y)$ and $\eta \equiv \eta(x, y)$ (which are known as characteristic equation also) then the standard form (which is known as canonical form too) of the partial differential equation (1) is as follows :

(i) $u_{\xi\xi} + u_{\eta\eta} = f(\xi, \eta, u, u_\xi, u_\eta)$ if the partial differential equation (1) is elliptic form [i.e. if $\Delta(x, y) < 0$]

(ii) $u_{\xi\xi} = f(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\eta\eta} = f(\xi, \eta, u, u_\xi, u_\eta)$, if the partial differential equation (1) is parabolic type [i.e., if $\Delta(x, y) = 0$]

(iii) $u_{\xi\xi} - u_{\eta\eta} = f(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\xi\eta} = f(\xi, \eta, u, u_\xi, u_\eta)$ if the partial differential equation (1) is hyperbolic type [i.e., if $\Delta(x, y) > 0$].

[Here $f \equiv f(\xi, \eta, u, u_\xi, u_\eta)$ is a function of the independent variables ξ, η, dependent variable u and their first order partial derivatives u_ξ and u_η.

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To reduce (1) into its corresponding canonical form, we use the following computations [using $\xi = \xi(x, y)$, $\eta = \eta(x, y)$]

We have $u = u(x, y)$

So, $u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_x\xi_x + u_\eta \eta_x$

Similarly, $u_y = u_\xi \xi_y + u_\eta \eta_y$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}$$

$$u_{yy} = u_{\xi\xi}\xi_\eta^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}$$

and $u_{xy} = u_{\xi\xi}\xi_x\xi_\eta + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_\eta\eta_x\eta_{xy} + u_\xi\xi_{xy} + u_\eta\eta_{xy}$.

With these transformations (1) reduces to

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = g \qquad \text{... (2)}$$

where the co-efficients of (2) are :

$$a \equiv a(\xi, \eta) = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$b \equiv b(\xi, \eta) = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$c \equiv c(\xi, \eta) = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$d \equiv d(\xi, \eta) = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$$e \equiv e(\xi, \eta) = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$$f = f(\xi, \eta) = F$$

$$g = g(\xi, \eta) = G$$

Further it can be verified that both of (1) and (2) have same nature, since $b^2 - 4ac = (\xi_x\eta_y - \xi_y\eta_x)^2(B^2 - 4AC)$

Below we discuss the reduction procedures of a standard partial differential equation [i.e. of the form (1)] to its corresponding canonical form :

■ Canonical form for hyperbolic partial differential equation

To transform a partial differential equation from its standard form (1) to the canonical form we must have $\Delta(x, y) = [B(x, y)]^2 - 4A(x, y)\,C(x, y) > 0$.

To determine the coordinates $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ we can use $b^2 - 4ac > 0$ *i.e.* $[b(\xi, \eta)]^2 - 4a(\xi, \eta)\,c(\xi, \eta) > 0$. Since we are describing here on real co-ordinate system only, so to satify $b^2 - 4ac > 0$, we may choose $a(\xi, \eta) = 0$ and $c(\xi, \eta) = 0$.

Or, in other words to obtain $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ we may choose,

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

and $c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$ ... (3)

From (3) and (4) we may write ... (4)

$$\left(\frac{\xi_x}{\xi_y}\right)^2 A + \left(\frac{\xi_x}{\xi_y}\right)B + C = 0 \text{ and } \left(\frac{\eta_x}{\eta_y}\right)^2 A + \left(\frac{\eta_x}{\eta_y}\right)B + C = 0 \qquad \text{... (5)}$$

From (5), we may conclude that $\frac{\xi_x}{\xi_y}$ and $\frac{\eta_x}{\eta_y}$ are two roots of the quadratic equation

$$Ax^2 + Bx + C = 0$$

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Since the roots of the above equation $Ax^2 + Bx + C = 0$ are $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

therefore let us agree to write,

$$\frac{\xi_x}{\xi_y} = \frac{-B(x,y) + \sqrt{[B(x,y)]^2 - 4A(x,y)\,C(x,y)}}{2A(x,y)} = \phi(x, y) \quad \text{... (6)}$$

and
$$\frac{\eta_x}{\eta_y} = \frac{-B(x,y) + \sqrt{[B(x,y)]^2 - 4A(x,y)\,C(x,y)}}{2A(x,y)} = \psi(x, y) \quad \text{... (7)}$$

Further here $\xi \equiv \xi(x, y) = c_1$ and $\eta \equiv \eta(x, y) = c_2$ both are real and then from the relation $\xi = \xi(x, y) = c_1$, we have

$$d\xi = \xi_x dx + \xi_y dy = 0 \text{ giving } \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} \quad \text{... (8)}$$

From (6) and (8) we have, $\dfrac{dy}{dx} = -\phi(x, y)$ ... (9)

Similarly for $\eta = \eta(x, y)$, we have, $\dfrac{dy}{dx} = -\psi(x, y)$ (10)

clearly, after integration we may easily obtain both $\xi(x, y)$ and $\eta(x, y)$ and using those we may get the canonical form of the differential equation. After obtaining that canonical form, the differential equation can be solved easily. This phenomenon will be illustrated in the following example :

Illustration-1 : Reduce the following second order differential equation into its canonical form and hence solve it : $2u_{xx} + 5u_{xy} + 3u_{yy} = 0$ ... (11)

■ Here the given differential equation is $2u_{xx} + 5u_{xy} + 3u_{yy} = 0$

Comparing (11) with its standard form

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$

we have $A = 2, B = 5, C = 3, D = E = F = G = 0$

Now $B^2 - 4AC = 25 - 24 = 1\ (> 0)$ and hence the given partial differential equation is of hyperbolic type. Thus to reduce (11) into its standard canonical form, we have to determine the transforming co-ordinates $\xi \equiv \xi(x, y)$ and $\eta \equiv \eta(x, y)$ which can be obtained from the following relation

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) = -\frac{-5+1}{4} = 1 \quad \text{... (12)}$$

$$\text{and } \frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right) = -\frac{-5-1}{4} = \frac{3}{2} \quad \text{... (13)}$$

From (12) and (13) we have respectively after integration

$y - x = c_1 = \xi(x, y)$ and $2y - 3x = c_2 = \eta(x, y)$.

Thus to reduce (11) into its canonical form the required transformations are :

$\xi(x, y) = y - x$ and $\eta(x, y) = 2y - 3x$

Then we have, $\xi_x = -1, \xi_y = 1, \xi_{xx} = \xi_{xy} = \xi_{yy} = 0$

and $\eta_x = -3, \eta_y = 2, \eta_{xx} = \eta_{xy} = \eta_{yy} = 0$

So if (11) can be reduced to the canonical from $au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = g$,

then $a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 2\cdot(1) + 5\cdot(-1)(1) + 3 = 0$

$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$= 4(-1)(-3) + 5[(-1) \times 2 + 1 \times (-3)] + 2 \times 3 \times 1 \times 2 = -1$

$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 2 \times (-3)^2 + 5 \times (-3) \times 2 + 3 \times (2)^2 = 30 - 30 = 0$

Similarly, $d = e = f = g = 0$

with these values the given partial differential equation (11) reduces to

$-u_{\xi\eta} = 0$ *i.e.*, $u_{\xi\eta} = 0$ ... (12)

On integration, we have, $u(\xi, \eta) = f_1(\xi) + f_2(\eta)$, where f_1 and f_2 are two arbitrary functions.

i.e., the required general salution is : $u(x, y) = f_1(y - x) + f_2(2y - 3x)$ ... (13)

Therefore the required canonical form is given by equation (12) and (13) gives its general solution.

■ Canonical form for parabolic partial differential equation

To transform a parabolic type partial differential equation from its standard form (1) to its canonical form we have both $[B(x, y)]^2 - 4A(x, y)C(x, y) = [b(\xi, \eta)]^2 - 4a(\xi, \eta)c(\xi, \eta) = 0$ upon suitable chosen ξ and η. Now to determine ξ and η we may set $b(\xi, \eta)$ and either of $a(\xi, \eta)$ or $c(\xi, \eta) = 0$

Now setting $a(\xi, \eta) = 0$, we have, $a(\xi, \eta) = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$

$$\text{i.e., } A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

which ensures that $\frac{\xi_x}{\xi_y}$ is a root of the equation $Ax^2 + Bx + C = 0$

$$\text{i.e., } \frac{\xi_x}{\xi_y} = -\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = -\frac{B}{2A}, \text{ since } B^2 - 4AC = 0$$

Again from the relation $\xi = \xi(x, y) = c_1$ = constant, we have

$$d\xi = \xi_x dx + \xi_y dy = 0 \text{ giving } \frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$$

Comparing above two we have, $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A}$ ... (14)

On integration (14), we may obtain the value for suitable transformation $\xi \equiv \xi(x, y)$ and the other co-ordinate $\eta \equiv \eta(x, y)$ should be chosen on such a way that $\eta(x, y)$ is not parallel with $\xi(x, y)$ (*i.e.*, the Jacobian, $J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_x \end{vmatrix} \neq 0$)

Note

1. In fact here instead of determining $\xi \equiv \xi(x, y)$, we at first can determine $\eta \equiv \eta(x, y)$ by setting $C = 0$ and from the relation $\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B}{2A}$ and then we may choose $\xi = \xi(x, y)$, such that ξ is not parallel to $\eta(x, y)$.
2. It is easy to show that for the value of $\xi(x, y)$ determining from the equation (14), $b(\xi, \eta)$ would also vanish. On the other hand if we determine $\eta(x, y)$ first from the relation $\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B}{2A}$, then also $b(\xi, \eta)$ vanishes.

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◘ Illustration-2 : Reduce the partial differential equation $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$ to its canonical form.

■ Here the given partial differential equation is

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0 \quad \text{... (15)}$$

Comparing (15) with its standard from $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we have $A = x^2$, $B = 2xy$, $C = y^2$, $D = E = F = G = 0$

Now we have $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$

Hence the given partial differential equation is of parabolic type.

To determine the canonical form of (15), we choose the co-ordinate $\xi(x, y)$ from the following equation :

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{2xy}{2x^2} = \frac{y}{x}, \text{ i.e., } \frac{dy}{y} - \frac{dx}{x} = 0$$

On integration we have $\frac{x}{y} = C_1$ and thus let $\xi = \frac{x}{y}$ be one independent co-ordinate of the canonical form. To determine other independent co-ordiante $\eta = \eta(x, y)$, it should be chosen in such a way that the Jacobian $\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$, let us choose $\eta = x$. Then we have

$$\xi_x = \frac{1}{y}, \xi_y = -\frac{x}{y^2}, \xi_{xx} = 0, \ \xi_{xy} = -\frac{1}{y^2}, \xi_{yy} = \frac{2x}{y^3} \text{ and } \eta_x = 1, \eta_y = 0 = \eta_{xx} = \eta_{xy} = \eta_{yy}$$

Now if the canonical form of partial differential equation (15) is $au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = g$, then the co-efficients can be computed as follows

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = x^2\left(\frac{1}{y^2}\right) + 2xy\left(\frac{1}{y}\right)\left(-\frac{x}{y^2}\right) + y^2.\left(\frac{x^2}{y^4}\right) = 0$$

$$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 2x^2\left(\frac{1}{y}\right)\cdot 1 + 2xy\left(\frac{1}{y}\right)\times 0 + 2xy\left(-\frac{x}{y^2}\right)\cdot 1 + 0 = 0$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = x^2$$

$$d = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = x^2 \times 0 + 2xy\left(-\frac{1}{y^2}\right) + y^2\left(\frac{2x}{y^3}\right) + 0 = 0$$

$$e = A\eta_x + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0$$

and $f = g = 0$

Therefore the canonical form of the partial differential equation (15) is given by

$$x^2u_{\eta\eta} = 0 \text{ i.e., } u_{\eta\eta} = 0 \quad \text{... (16)}$$

Clearly equation (16) is the required canonical form of the partial differential equation (15) and its solution is $u(\xi, \eta) = \eta f_1(\xi) + f_2(\xi)$ i.e., $u(x, y) = yf_1\left(\frac{x}{y}\right) + f_2\left(\frac{x}{y}\right)$ where f_1 and f_2 are two arbitrary functions of x and y.

Note

It should be noted that representation in canonical form of every second order partial differential equation is possible but its solution always may not be possible.

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Canonical form of Elliptic partial differential equation

To transform an elliptic type partial differential equation from its standard form (1) to its canonical form, we must have, the discriminant $\Delta(x, y) = \{B(x, y)\}^2 - A(x, y)\, C(x, y) < 0$. Now if the Cartesian coordinates (x, y) transforms to another coordinate system (ξ, η), then from the relation $\xi \equiv \xi(x, y) = C_1$ and $\eta \equiv \eta(x, y) = C_2$ [*i.e.*, from $d\xi = \xi_x dx + \xi_y dy = 0$ and $d\eta = \eta_x dx + \eta_y dy = 0$] we have

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = -\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right) = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

In this case since $\Delta(x, y) = B^2 - 4AC < 0$, then after integration we will have both of $\xi(x, y)$ and $\eta(x, y)$ as complex conjugate function of x and y. To obtain real transformations, we need another adjustments by choosing (ρ, ζ) transformation as a new transformation as follows : $\rho = \frac{\xi + \eta}{2}$ and $\zeta = \frac{\xi - \eta}{2i}$. Using these transformations, finally a partial differential equation reduces to the canonical form which is as follows

$$u_{\rho\rho} + u_{\zeta\zeta} = f(u_p, u_\xi, u, p, \xi) \quad \text{... (17)}$$

We illustrate this type of transformation in the following example.

Illustration-3 : Reduce the partial differential equation $y^2 u_{xx} + u_{yy} = 0$ into its canonical form.

Here the given partial differential equation is

$$y^2 u_{xx} + u_{yy} = 0 \quad \text{... (18)}$$

Comparing (18) with its standard form (1), *i.e.*,

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ we have, $A = y^2$, $C = 1$, $B = 0 = D = E = F = G$

Here the discriminant of (18) is $B^2 - 4AC = -4y^2 < 0$.

Thus the equation (18) is of elliptic type equation.

To obtain the canonical form of (18), we use the transformations

$\xi \equiv \xi(x, y)$ and $\eta \equiv \eta(x, y)$ which can be obtained from

$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\xi_x}{\xi_y}$ and $\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = -\frac{\eta_x}{\eta_y}$, which gives $\frac{dy}{dx} = \mp\frac{i}{y}$, since $B^2 - 4AC = -4y^2$ and $B = 0$.

On integration, we have, $y^2 = -2ix + C_1$ and $y^2 = 2ix + C_2$ giving $C_1 = \xi(x, y) = 2ix + y^2$ and $C_2 = \eta(x, y) = -2ix + y^2$, so that we may choose the new real co-ordinate system as

$$\rho(x, y) = \frac{\xi(x, y) + \eta(x, y)}{2} = y^2 \text{ and } \zeta(x, y) = \frac{\xi(x, y) - \eta(x, y)}{2} = 2x$$

Then we have, $\rho_x = 0 = \rho_{xy} = \rho_{xx}$ and $\rho_y = 2y$, $\rho_{yy} = 2$ and $\zeta_x = 2$, $\xi_{xx} = \xi_{xy} = \xi_y = \xi_{yy} = 0$.

Thus if (18) can be transformed to its equivalent canonical form

$au_{\rho\rho} + bu_{\rho\zeta} + cu_{\zeta\zeta} + du_\rho + eu_\zeta + fu = g$, then we have

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$a = A\rho_x^2 + B\rho_x\rho_y + C\rho_y^2 = 4y^2$

$b = 2A\rho_x\zeta_x + B(\rho_x\zeta_y + \rho_y\zeta_x) + 2C\rho_y\zeta_y = 0$

$c = A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 4y^2$

$d = A\rho_{xx} + B\rho_{xy} + C\rho_{yy} + D\rho_x + E\rho_y = 2$

$e = A\zeta_{xx} + B\zeta_{xy} + C\zeta_{yy} + D\zeta_x + E\zeta_y = 0$

$f = 0 = g.$

Therefore the canonical form of the partial differential equation (18) becomes as follows

$$4y^2u_{\rho\rho} + 4y^2u_{\zeta\zeta} + 2u_\rho = 0 \ \text{ i.e., } u_{\rho\rho} + u_{\zeta\zeta} = -\frac{u_\rho}{2\rho} \qquad \text{... (19)}$$

since $\rho = y^2$

Hence the required canonical transformation of the elliptic type partial differential equation (18) is given by the equation (19), where $\rho \equiv \rho(x, y) = y^2$ and $\zeta \equiv \zeta(x, y) = 2x$.

Illustrative Examples

1. **Reduce the equation to its corresponding canonical form and hence solve it, if possible : $xu_{xx} + (x + y)u_{xy} + yu_{yy} = 0$**

Solution Here, the given second order partial differential equation is

$$xu_{xx} + (x + y)u_{xy} + yu_{yy} = 0 \qquad \text{... (1)}$$

Comparing (1) with its general form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad \text{... (2)}$$

we have, $A = x$, $B = (x + y)$, $C = y$, $D = E = F = G = 0$

Now the discriminant of the equation (1) is

$$\Delta(x, y) \equiv B^2 - 4AC = (x + y)^2 - 4xy = (x - y)^2 > 0$$

Clearly the discriminant of (1) is positive except the straight line $y = x$ and since it is the trivial case, then we may omit this case. So except the straight line $x = y$, the canonical transformations $\xi(x, y)$ and $\eta(x, y)$ can be determined from the relations

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{(x + y) - (x - y)}{2x} = \frac{y}{x} \qquad \text{... (3)}$$

$$\text{and } \frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{(x + y) + (x - y)}{2x} = 1 \qquad \text{... (4)}$$

Integrating (3) and (4) we have the required transformation,

$$C_1 = \xi(x, y) = \frac{x}{y} \text{ and } C_2 = \eta(x, y) = y - x$$

For these two transformations, we have,

$$\xi_x = \frac{1}{y}, \xi_y = -\frac{x}{y^2}, \xi_{xx} = 0, \xi_{xy} = -\frac{1}{y^2}, \xi_{yy} = \frac{2x}{y^3} \text{ and}$$

$$\eta_x = -1, \eta_y = 1, \eta_{xx} = \eta_{xy} = \eta_{yy} = 0$$

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If the canonical form of (1) is as follows :

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = g \quad \text{... (5)}$$

then the coefficients can be determined from

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = x\cdot\frac{1}{y^2} + (x+y)\cdot\frac{1}{y}\left(-\frac{x}{y^2}\right) + y\cdot\frac{x^2}{y^4} = \frac{x}{y^2} - \frac{x^2}{y^3} - \frac{x}{y^2} + \frac{x^2}{y^3} = 0$$

$$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 2x\cdot\frac{1}{y}(-1) + (x+y)\left[\frac{1}{y}.1 - \frac{x}{y^2}(-1)\right] + 2y\left(-\frac{x}{y^2}\right).1$$

$$= -\frac{2x}{y} + \frac{1}{y^2}(x+y)^2 - \frac{2x}{y} = \frac{1}{y^2}(x-y)^2$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = x + (x+y)(-1)(1) + y = 0$$

$$d = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = x\cdot 0 + (x+y)\left(-\frac{1}{y^2}\right) + y\cdot\frac{2x}{y^3} = \frac{x}{y^2} - \frac{1}{y} = \frac{1}{y^2}(x-y)$$

$$e = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0$$

$$f = g = 0$$

Using above results in (5), the required canonical transformation becomes as follows :

$$\frac{1}{y^2}(x-y)^2u_{\xi\eta} + \frac{1}{y^2}(x-y)u_\xi = 0$$

i.e., $(x-y)u_{\xi\eta} + u_\xi = 0$ [Assuming $y \neq 0$ and $x \neq y$]

i.e., $-\eta u_{\xi\eta} + u_\xi = 0$, since $\eta(x, y) = y - x$

i.e., $u_{\xi\eta} = \frac{1}{\eta}u_\xi$ ... (6)

Equation (6) is the required canonical form of the given partial differential equation (1).

Now from (6) we have $\frac{\partial}{\partial\eta}\left(\frac{\frac{\partial u}{\partial\xi}}{\eta}\right) = \frac{\eta u_{\xi\eta} - u_\xi}{\eta^2} = 0$

and so integrating we have, $\frac{1}{\eta}\frac{\partial u}{\partial\xi} = f_1(\xi)$, where f_1 is an artibrary function of ξ

i.e., $\frac{\partial u}{\partial\xi} = \eta f_1(\xi)$

Again integrating we have,

$$u(\xi, \eta) = \int\xi f_1(\xi)d\xi + f_2(\eta), \text{ where } f_2 \text{ is another arbitrary function}$$

$$= \eta\int f_1(\xi)d\xi + f_2(\eta)$$

i.e., $u(x, y) = (y-x)\int f_1\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right) + f_2(y-x)$, which is the required general solution for f_1 and f_2 being two arbitrary functions.

2. **Find the characteristic equations to reduce the partial differential equation $u_{xx} + 2u_{xy} + \cos^2 x\, u_{yy} + 2u_x + u_y = 0$ into its canonical form.**

Solution Here the given partial differential equation is

$$u_{xx} + 2u_{xy} + \cos^2 x\, u_{yy} + 2u_x + u_y = 0 \quad \text{... (1)}$$

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Comparing (1) with its general form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \qquad \text{... (2)}$$

we have $A = 1, B = 2, C = \cos^2 x, D = 2, E = 1, F = G = 0$

Therefore the discriminant of (1) is

$$\Delta(x, y) = B^2 - 4AC = 4 - 4\cos^2 x = 4\sin^2 x > 0 \text{ for } x \neq n\pi, n \in Z$$

Thus the given partial differential equation is of parabolic type.

Now to reduce (1) into its canonical form the transformations required as $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ where ξ and η can be determined from the following differential equations

$$\frac{dy}{dx} = -\left\{\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right\} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = \frac{2 \mp \sqrt{4 - 4\cos^2 x}}{2} = 1 \mp \sin x$$

Thus the characteristic equations of (1) are given by solving the differential equatios

$$\frac{dy}{dx} = 1 - \sin x \text{ and } \frac{dy}{dx} = 1 + \sin x \quad i.e., y + C_1 = x + \cos x \text{ and } y + C_2 = x - \cos x$$

Hence the characteristic equation are given by

$$C_1 = \xi(x, y) = x + \cos x - y \text{ and } C_2 = \eta(x, y) = x - \cos x - y$$

3. **Classify and transform the following equation to its responding canonical form : $\cos^2 x\, u_{xx} + \sin 2x\, u_{xy} + \sin^2 x\, u_{yy} - \cot x\, u_x = y^2$.**

Solution Here the given partial differential equation is

$$\cos^2 x\, u_{xx} + \sin 2x\, u_{xy} + \sin^2 x\, u_{yy} - \cot x\, u_x = y^2 \qquad \text{... (1)}$$

Comparing (1) with its general form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad \text{... (2)}$$

We have, $A = \cos^2 x, B = \sin 2x, C = \sin^2 x, D = -\cot x, E = F = 0, G = y^2$

Now the discriminant of (2) is $\Delta(x, y) = B^2 - 4AC = \sin^2 2x - 4\sin^2 x \cos^2 x = 0$

Then the given partial differential equation is of parabolic type.

To reduce (1) into its canonical form, we need to determine the transformations $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

Since the partial differential equation (1) is of parabolic type, then the transformation coordinate $\xi(x, y)$ can be determined from the expression $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A}$

i.e., here $\frac{dy}{dx} = \frac{\sin 2x}{2\cos^2 x} = \tan x$

Integrating we have, $y + \log\cos x = C_1$

Thus the coordinate $\xi(x, y)$ can be determined from, $\xi(x, y) = y + \log\cos x$.

Thus the coordinate $\xi(x, y)$ can be determined in such a way that the

So, the other independent coordinates $\eta(x, y)$ can be determined in such a way that the

Jacobian $\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$, so that we may choose $\eta(x, y) = x$.

Now $\xi_x = -\tan x, \xi_{xx} = -\sec^2 x, \xi_{xy} = 0, \xi_y = 1, \xi_{yy} = 0$

$\eta_x = 1, \eta_y = 0, \eta_{xx} = 0, \eta_{xy} = 0, \eta_{yy} = 0$

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Now if the canonical form of (1) is $au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = g$, where the coefficients are determined as,

$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = \cos^2x\tan^2x + \sin2x(-\tan x)(1) + \sin^2x = 2\sin^2x - 2\sin^2x = 0$

$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$= 2\cos^2x(-\tan x)(1) + \sin2x(-\tan x(0) + 1\cdot1) + 2\sin^2x(-1)\times0$

$= -2\sin x\cos x + 2\sin x\cos x = 0$

$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = \cos^2x + \sin2x(1)(0) + \sin^2x\times0 = \cos^2x$

$d = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$

$= \cos^2x(-\sec^2x) + \sin2x\times0 + \sin^2x\times0 + (-\tan x)(-\cot x) = 0$

$e = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = (-\cot x)\cdot1 = -\cot x$

$f = 0, g = y^2$

Thus the canonical form of (1) can be written as

$$\cos^2x\,u_{\eta\eta} - \cot x\,u_\eta = y^2$$

i.e., $\cos^2\eta\,u_{\eta\eta} - \cot\eta\,u_\eta = (\xi - \log\cos\eta)$

i.e., $u_{\eta\eta} = \cot\eta\sec^2\eta\,u_\eta + \sec^2\eta\,(\xi - \log\cos\eta)$

i.e., $u_{\eta\eta} = \dfrac{\sec\eta}{\sin\eta}u_\eta + (\xi - \log\cos\eta)\sec^2\eta$ ... (3)

Hence (3) is the required canonical form of the given partial differential equation (1)

4. Reduce the following partial differential equation $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$ to its equivalent canonical form.

Solution Here the given partial differential equation is

$$(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0 \quad ... (1)$$

Comparing (1) with its general form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad ... (2)$$

we have $A = 1 + x^2$, $B = 0$, $C = 1 + y^2$, $D = x$, $E = y$, $F = G = 0$.

The discriminant of (1) is thus $\Delta(x, y) = B^2 - 4AC = -4(1 + x^2)(1 + y^2) < 0$ for all real values of x and y.

Thus the given partial differential equation is of elliptic type partial differential equation.

To determine the canonical form of (1) we have to first find the characteristic equations.

If $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ be the new coordinates, then ξ and η can be determined from

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{-\sqrt{-4(1+x^2)(1+y^2)}}{2(1+x^2)} = -i\sqrt{\frac{1+y^2}{1+x^2}}$$

and $$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{\sqrt{-4(1+x^2)(1+y^2)}}{2(1+x^2)} = i\sqrt{\frac{1+y^2}{1+x^2}}$$

Solving above two differential equations, we have,

$$\log(x + \sqrt{x^2+1}) - i\log(y + \sqrt{y^2+1}) = C_1 = \xi(x, y)$$

and $$\log(x + \sqrt{x^2+1}) + i\log(y + \sqrt{y^2+1}) = C_2 = \eta(x, y)$$

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Since here both ξ and η are in complex conjugate from, When we need to introduce another transformation which is defined as below :

$$\rho(x, y) = \frac{\xi + \eta}{2} \text{ and } \zeta(x, y) = \frac{\eta - \xi}{2i},$$

so that $\rho(x, y) = \log(x + \sqrt{x^2+1})$ and $\zeta(x, y) = \log(y + \sqrt{y^2+1})$

Now, we have $\rho_x = \frac{1}{\sqrt{x^2+1}}$, $\rho_y = 0$, $\rho_{xx} = -\frac{x}{(x^2+1)\sqrt{x^2+1}}$, $\rho_{xy} = \rho_{yy} = 0$

and $\zeta_x = 0$, $\zeta_y = \frac{1}{\sqrt{y^2+1}}$, $\zeta_{xx} = 0$, $\zeta_{xy} = 0$, $\zeta_{yy} = -\frac{y}{(y^2+1)\sqrt{y^2+1}}$

Now, if the canonical form of the given equation (1) is

$$a u_{\rho\rho} + b u_{\rho\zeta} + c u_{\zeta\zeta} + d u_\rho + e u_\zeta + fu = g \quad \dots (3)$$

then the coefficients are computed as follows

$a = A\rho_x^2 + B\rho_x\rho_y + C\rho_y^2 = (1 + x^2)\cdot\frac{1}{x^2+1} = 1$

$b = 2A\rho_x\zeta_x + B(\rho_x\zeta_y + \rho_y\zeta_x) + 2C\rho_y\zeta_y$

$= 2(1 + x^2)\cdot\frac{1}{\sqrt{1+x^2}} \times 0 + 0 + 2(1 + y^2)\cdot\frac{1}{\sqrt{1+y^2}} \times 0 = 0$

$c = A\zeta_x^2 + B\zeta_x\zeta_x + C\zeta_y^2 = (1 + y^2)\cdot\frac{1}{1+y^2} = 1$

$d = A\rho_{xx} + B\rho_{xy} + C\rho_{yy} + D\rho_x + E\rho_y = (1 + x^2)\left(-\frac{x}{(x^2+1)\sqrt{1+x^2}}\right) + \frac{x}{\sqrt{1+x^2}} = 0$

$e = A\zeta_{xx} + B\zeta_{xy} + C\zeta_{yy} + D\zeta_x + E\zeta_y = (1 + y^2)\left(-\frac{y}{(1+y^2)\sqrt{1+y^2}}\right) + \frac{y}{\sqrt{1+y^2}} = 0$

and $f = 0, g = 0$

Using above results in (3), required canonical transformation of the given partial differential equation is given by $u_{\rho\rho} + u_{\zeta\zeta} = 0$ ... (4)

where $\rho \equiv \rho(x, y) = \log(x + \sqrt{x^2+1})$ and $\zeta \equiv \zeta(x, y) = \log(y + \sqrt{y^2+1})$.

Exercise

- **Reduce the following partial differential equations to corresponding canonical form and hence solve them, if possible.**

1. $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} - \frac{y^2}{x} u_x - \frac{x^2}{y} u_y = 0$
2. $\sec x\, u_{xx} - 2\tan x\, u_{xy} - \cos x\, u_{yy} = u_y$
3. $a^2 x u_{xx} - x u_{yy} + 2b u_x = 0$
4. $u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0$
5. (a) The Tricomi equation $u_{xx} + x u_{yy} = 0$ for $x < 0$

   (b) The Tricomi equation $u_{xx} + x u_{yy} = 0$ for $x > 0$.

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Answers

1. The given partial differential equation is of parabolic type and the canonical form is $u_{\eta\eta} = 0$ and general solution is $u(x, y) = y^2 f_1(x^2 + y^2) + f_2(x^2 + y^2)$

2. The given partial differential equation is of hyperbolic type and the canonical from is $u_{\xi\eta} = 0$. The general solution is $u(x, y) = f_1(x - y + \cot x) + f_2(x - \cot x + y)$.

3. Parabolic type equation and canonical form is $u_{\xi\eta} = -\dfrac{b}{\xi + \eta}(u_\xi + u_\eta)$ where $\xi = x + ay$ and $\eta = x - ay$.

4. Elliptic type equation and the canonical form is

$$u_{\rho\rho} + u_{\zeta\zeta} = \frac{1}{3}u_\rho - \frac{2}{\sqrt{3}}u_\zeta,$$

where $\rho = y - x$, $\zeta = \sqrt{3}\,x$.

5. (a) Hyperbolic type and the canonical form is $u_{\xi\eta} = \dfrac{u_\xi - u_\eta}{6(\xi - \eta)}$ where $\xi(x, y) = \dfrac{3}{2}y - \left(\sqrt{-x}\right)^3$ and $\eta(x, y) = \dfrac{3}{2}y + \left(\sqrt{-x}\right)^3$ [since $x < 0$]

(b) Elliptic type and the canonical form is $u_{\rho\rho} + u_{\zeta\zeta} = -\dfrac{u_\zeta}{3\zeta}$, where $\rho = \dfrac{3}{2}y$ and $\zeta = -\left(\sqrt{x}\right)^3$.

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Multiple Choice Questions (MCQ)

1. The C.F. of the differential equation $(D^2 - 5D + 6)y = e^{2x} \cdot x^3$ is

   a. $Ae^{2x} + Be^{3x}$ b. $(A + Bx)e^{2x}$

   c. $Ae^{-2x} + Be^{3x}$ d. $Ae^{-2x} + Be^{-3x}$

2. The C.F. of the differential equation $(9D^2 + 12D + 4)y = \cos 7x$ is

   a. $(A + Bx)e^{\frac{2}{3}x}$ b. $(A + Bx)e^{-\frac{2}{3}x}$

   c. $Ae^{\frac{2}{3}x} + Be^{-\frac{2}{3}x}$ d. None of these

3. The general solution of the differential equation $(D^3 + 3D^2 + 3D + 1)y = 0$ is

   a. $(A + Bx + Cx^2)e^{-x}$

   b. $(A - Bx + Cx^2)e^{x}$

   c. $(A + Bx)e^{-x}$

   d. $(A + Bx)e^{x}$

4. The particular integral of the differential equation $(D^2 + 1)y = \cos 2x$ is

   a. $\frac{1}{3}\cos 2x$ b. $-\frac{1}{3}\cos 2x$

   c. $3\cos 2x$ d. $\frac{1}{5}\cos 2x$

5. The particular integral of the differential equation $(D^2 + a^2)y = \cos ax$ is

   a. $\frac{x\cos ax}{2a}$ b. $\frac{x\sin ax}{2}$

   c. $\frac{x\sin ax}{2a}$ d. $-\frac{x\sin ax}{2a}$

6. $\frac{1}{D-1}x^2 =$

   a. $x^2 + 2x + 2$ b. $-(x^2 + 2x + 2)$

   c. $2x - x^2$ d. $x^2 - 2x$

7. The particular integral of $(D^2 - 4D + 4)y = x^3e^{2x}$ is

   a. $\frac{e^{2x} \cdot x^4}{20}$ b. $\frac{e^{2x} \cdot x^5}{20}$

   c. $\frac{e^{2x} \cdot x^4}{60}$ d. $\frac{e^{x} \cdot x^5}{20}$

8. The general solution of the differential equation $(D + 1)^3(D^2 + 1) = 0$ is

   a. $y = (C_1 + C_2x + C_3x^2)e^{-x} + C_4\cos x + C_5\sin x$

   b. $y = (C_1 + C_2x)e^{-x} + C_3\cos x + C_4\sin x$

   c. $y = (C_1 + C_2x + C_3x^2)e^{-x}$

   d. None of these

9. Using the substitution $x = e^z$, equation $x^2\frac{d^2y}{dx^2} - 5y = \log x$ reduces to

   a. $\frac{d^2y}{dz^2} + \frac{dy}{dz} - 5y = z$

   b. $\frac{d^2y}{dz^2} - \frac{dy}{dz} - 5y = z$

   c. $\frac{d^2y}{dz^2} - 2\frac{dy}{dz} - 5y = z$

   d. None of these

10. The C.F. of the differential equation $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 6x^5$ is

    a. $Ax + Bx^2$ b. $Ax^2 + Bx^3$

    c. $Ae^{2x} + Be^{3x}$ d. $Ae^{-2x} + Be^{-3x}$

11. The particular integral of the differential equation $(x^2D^2 - xD + 1)y = 2\log x$ is

    a. $2\log x$ b. $2e^z + 4$

    c. $2\log x + 4$ d. $2\log x - 4$

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12. The auxiliary equation of the differential equation $(x + a)^2 \frac{d^2y}{dx^2} - 5(x + a)\frac{dy}{dx} + 6y = 0$ is

a. $m^2 + 6m + 6 = 0$

b. $m^2 + 6m - 5 = 0$

c. $m^2 - 6m + 6 = 0$

d. $m^2 - 6m - 6 = 0$

13. From the system of equations $\frac{dx}{dt} + 5x + y = e^t$, $\frac{dy}{dt} - x + 3y = e^{2t}$ the obtained differential equation of the two variables x and t is

a. $\frac{d^2x}{dt^2} - 8\frac{dx}{dt} + 16x = 4e^t - e^{2t}$

b. $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 4e^t + e^{2t}$

c. $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 4e^t - e^{2t}$

d. None of these

14. The differential equation involving y and t as obtained from $\frac{dx}{dt} = 3x + 8y$, $\frac{dy}{dt} = -x - 3y$ is

a. $\frac{d^2y}{dt^2} + y = 0$ b. $\frac{d^2y}{dt^2} - y = 0$

c. $\frac{d^2y}{dt^2} + 5y = 0$ d. $\frac{d^2y}{dt^2} + 3y = 0$

15. From the simultaneous equation $\frac{dx}{dt} - 5x - 4y = 0$, $\frac{dy}{dt} + x - y = 0$, we get

a. $y = (A + Bt)e^{-3t}$

b. $y = Ae^{3t} + Be^{-3t}$

c. $y = (A + Bt)e^{3t}$

d. None of these

16. From the simultaneous equation $\frac{dx}{dt} - 3x - 2y = 0$, $\frac{dy}{dt} - 5x - 3y = 0$, we get

a. $x = A\cos 2t + B\sin 2t$

b. $x = A\cos t + B\sin t$

c. $x = (A + Bt)e^{2t}$

d. None of these

17. The differential equation obtained by elimination arbitrary constants a and b from the equation $z = ax + by + ab$ is

a. $\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = 0$ b. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

c. $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$ d. None of these

18. The differential equation of all spheres of radius λ, having centre in the xy plane is

a. $z\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = \lambda$

b. $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = \lambda^2$

c. $z\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 - 1\right] = \lambda^2$

d. None of these

19. Order and degree of the differential equation $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = 1$ are respectively

a. 1 and 1 b. 2 and 1

c. 1 and 2 d. None of these

20. Equiation $\frac{\partial^2 z}{\partial x^2} - 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial y}\right)^2 = 0$ is of order

a. 1 b. 2

c. 3 d. None of these

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21. The Lagrange's auxiliary equations for the partial differential equation $Pp + Qq = R$ are

a. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ b. $\frac{dx}{P} = \frac{dy}{Q}$

c. $\frac{dx}{P} = \frac{dz}{R}$ d. None of these

22. The Lagrange's auxiliary equation for the partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ are

a. $\frac{dx}{x(y^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$

b. $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$

c. $\frac{dx}{y(y^2+z)} = \frac{dy}{x(y^2+z)} = \frac{dz}{(x^2-y^2)z}$

d. None of these

23. The general solution of the linear partial differential equation $Pp + Qq = R$ is

a. $f(u, v) = 0$ b. $f(u, v) = -1$

c. $f(u, v) = 1$ d. None of these

24. Charpit's auxiliary equation of the partial differential equation $z = pq$ are

a. $\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{-2pq} = \frac{dp}{-p} = \frac{dq}{-q}$

b. $\frac{dx}{p} = \frac{dy}{-q} = \frac{dz}{2pq} = \frac{dp}{p} = \frac{dq}{q}$

c. $\frac{dx}{-q} = \frac{dy}{-p} = \frac{dz}{-2pq} = \frac{dp}{-p} = \frac{dq}{-q}$

d. None of these

25. The complete integral of the partial differential equation $q = 3p^2$ is

a. $z = ax + 3by$

b. $z = ax + 3a^2y + b$

c. $z = 3a^2x + ay + b$

d. None of these

26. The particular integral of the differential equation $(D - 2)^2y = 6 \cdot e^{2x}$ is

a. x^2e^{2x} b. $\frac{x^2}{3} \cdot e^{2x}$

c. $3x \cdot e^x$ d. $3x^2 \cdot e^{2x}$

27. The differential equation involving y and t as obtained from $\frac{dx}{dt} + 2y = 0$, $\frac{dy}{dt} - 2y = x$ is

a. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$

b. $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$

c. $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 0$

d. None of these

28. The particular integral of $(D^2 + 9)y = 4\cos\left(x + \frac{\pi}{3}\right)$ is

a. $\frac{1}{2}\cos\left(x + \frac{\pi}{3}\right)$

b. $\frac{1}{3}\cos\left(x + \frac{\pi}{3}\right)$

c. $\frac{1}{4}\cos\left(x + \frac{\pi}{3}\right)$

d. $\frac{2}{3}\cos\left(x + \frac{\pi}{3}\right)$

29. The differential equation obtained by eliminating a and p from $z = ae^{pt} \cdot \sin px$ is

a. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0$ b. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$

c. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$ d. None of these

30. Equation $p\tan y + q\tan x = \sec^2 z$ is of order

a. 1 b. 2

c. 0 d. None of these

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31. The differential equations of all circles passing through the origin and having their centres on the x-axis is

a. $y^2 + x^2 = 2xy\frac{dy}{dx}$

b. $x^2 - y^2 = 2xy\frac{dy}{dx}$

c. $y^2 - x^2 = 2xy\frac{dy}{dx}$

d. $x^2 + 2xy\frac{dy}{dx} = 0$

32. The differential equation of all parabolas having their axes parallel to the y-axis is

a. $y_1 = 0$ b. $y_2 = 0$

c. $y_3 = 0$ d. $y_4 = 0$

33. The differential equation of all conics whose axes coincide with the axes of coordinates is

a. $xyy_2 + xy_1^2 = yy_1$

b. $xyy_2 - xy_1^2 + yy_1 = 0$

c. $xy_2 + xy_1^2 = yy_1$

d. none of these

34. The differential equation of the system of confocal conics $\frac{x^2}{a^2+P} + \frac{y^2}{b^2+P} = 1$ in which P is the parameter and a, b are given constant, is

a. $(a^2 - b^2)y_1 = x^2y_1 - xy$

b. $(a^2 - b^2) = \frac{x^2y_1 - xy}{y_1}$

c. $(a^2 + b^2 + x^2)y_1 = xy$

d. none of these

35. The differential equation from the relation $y^2 = A(B - x)(B + x)$, A, B are arbitrary constants is

a. linear b. non-linear

c. undefined d. none of these

36. The foci of the curve which satisfies the differential equation $(1 + y^2)dx - xydy = 0$ and passes through the point (1, 0) is

a. $(\pm 1, 0)$ b. $(\pm 2, 0)$

c. $(\pm \sqrt{2}, 0)$ d. $(\pm 3, 0)$

37. All curves for which the length of the normal is equal to the radius vector are

a. circles

b. rectangular hyperbolas

c. hyperbolas

d. both (a) and (b)

38. The angle between the radius vector and tangent is suplementary of half the vectorial angle. The equation of the curve is

a. $\frac{2c}{r} = 1 + \cos\theta$ b. $\frac{2c}{r} = 1 - \cos\theta$

c. $\frac{1}{r} = 1 - \cos\theta$ d. None of these

39. The integrating factor of $\cos^2 x\frac{dy}{dx} + y = \tan x$ is

a. $\tan x$ b. $e^{\tan x}$

c. $e^{\sec^2 x}$ d. $e^{\sec x}$

40. The general solution of $y = 2Px + y^2P^3$ is

a. $y^2 = 2cx + c^3$ b. $x^2 = 2cy + c^3$

c. $x^2 + y^2 = c$ d. none of these

41. The singular solution of

$P = \log(Px - y)\left(P = \frac{dy}{dx}\right)$ is

a. $x + y = x\log x$ b. $x(1 + \log x) = y$

c. $x - y = \log x$ d. $x + y = \log x$

42. The orthogonal trajectories of hyperbolas $xy = c^2$ is

a. $x^2 + y^2 = c$ b. $x^2 - y^2 = c$

c. $x + y = c$ d. $x - y = c$

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43. The equation $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$ satisfying the initial conditions $x = \frac{\pi}{2}, y = -4$. The required solution is

a. $y\sin x - 5e^{\cos x} = 1$

b. $y\sin x - e^{\cos x} = 1$

c. $y\sin x + 5e^{\cos x} = 1$

d. $y\sin x + e^{\cos x} = 1$

44. The particular integral of the differential equation $\left(\frac{d}{dx} - 2\right)^2 y = 6.e^{2x}$ is

a. $x^2 \cdot e^{2x}$ b. $\frac{1}{3}x^2 \cdot e^{2x}$

c. $3x \cdot e^x$ d. $3x^2 \cdot e^{2x}$

45. The particular integral of the differential equation $\frac{d^2y}{dx^2} + y = \cos 2x$ is

a. $\frac{1}{3}\cos 2x$ b. $-\frac{1}{3}\cos 2x$

c. $3\cos 2x$ d. $\frac{1}{5}\cos 2x$

46. The particular integral of the differential $\frac{d^8y}{dx^8} - 2\frac{d^4y}{dx^4} + y = \sin(2x+3)$ is

a. $\frac{1}{5}\sin(2x+3)$ b. $\frac{1}{125}\sin(2x+3)$

c. $\frac{1}{225}\sin(2x+3)$ d. $\frac{1}{625}\sin(2x+3)$

47. The particular integral of $(D^2 - 4D + 4)y = \cosh 2x + 3$ is

a. $\frac{x^2}{4}e^{2x} + \frac{e^{-2x}}{32}$ b. $\frac{e^{-2x}}{32} + \frac{3}{4}$

c. $\frac{x^2 e^{2x}}{4} + \frac{3}{4}$ d. $\frac{x^2}{4}e^{2x} + \frac{e^{-2x}}{32} + \frac{3}{4}$

48. The order and degree of the following differential equation $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 1 + x$ is [CP 1999]

a. order 1, degree 1

b. order 1, degree $\frac{1}{2}$

c. order 1, degree 2

d. order 2, degree 1

49. The order of the differential equation by eliminating the arbitrary constants A and B from the following function $y = A\cos x + B\sin x$ is [CP 1983, 1988]

a. 1 b. 2

c. 3 d. none of the above

50. If A and B are two arbitrary constants, then the order of the differential equation, formulated from the relation $y = A\cos(px - B)$, p being a parameter. [CP 2000]

a. 1 b. 2 c. 3 d. 4

51. Which of the following is an integrating factor of the differential equation $(2x^2 + y^2 + x)dx + xy\,dy = 0$

a. x b. y c. $\frac{1}{x}$ d. xy

52. The differential equation of all parabolas having their axes parallel to the y-axis is [CH 1994]

a. $y' = 0$ b. $y'' = 0$

c. $y''' = 0$ d. $y'''' = 0$

53. The differential equation by eliminating arbitrary constants A and B from $y = A\cos(\log x) + B\sin(\log x)$ is [NBH 2007]

a. $y^2\frac{d^2y}{dx^2} + y\frac{dy}{dx} + x = 0$

b. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = 0$

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c. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

d. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

54. **Find $\lim_{t\to\infty} x(t)$ where $x(t)$ can be obtained from $x + x = 0, x(0) = 2$ [BH 2003]**

a. 0 b. 1 c. ∞ d. e

55. **In the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 - y^2}$, after substituting $y = vx$, $x\frac{dv}{dx} =$**

a. $\frac{v^2}{1-v^2}$ b. $\frac{v^3}{1-v^2}$ c. $\frac{v}{1-v}$ d. $\frac{v}{1-v^2}$

56. **The differential equation $(x-y-1)dx + (4y+x-1)dy = 0$ can be reduced to a homogeneous differential equation using the transformation**

a. $u = x + 1, v = y$ b. $u = x, v = y - 1$

c. $y = x - 1, v = y$ d. $u = x - 1, v = y - 1$

57. **The integrating factor of the differential equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ is**

a. x^2 b. y^2 c. x^2y^2 d. $\frac{1}{x^2y^2}$

58. **The integrating factor of $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$ is**

a. $\frac{1}{x^3y^2}$ b. x^3y^3

c. $\frac{1}{2x^3y^3}$ d. none of the above

59. **The condition for integrating for the differential equation $p\,dx + q\,dy + r\,dz = 0$ is**

a. $p\left(\frac{\partial q}{\partial y} - \frac{\partial r}{\partial z}\right) + q\left(\frac{\partial r}{\partial z} - \frac{\partial p}{\partial x}\right) + r\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) = 0$

b. $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0$

c. $x\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) + y\left(\frac{\partial q}{\partial y} - \frac{\partial r}{\partial z}\right) + z\left(\frac{\partial r}{\partial z} - \frac{\partial p}{\partial x}\right) = 0$

d. none of the above

60. **The general solution of the differential equation $xy\,dz + yz\,dx + zx\,dy = 0$ is**

a. $x + y + z = 1$ b. $xyz = 1$

c. $x + y + z = c$ d. $xyz = c$

61. **If the condition $p\left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}\right) + q\left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) + r\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) = 0$, then which of the following differential equation must be solved?**

a. $p\,dx + q\,dy = 0$

b. $p\,dz + q\,dx + r\,dy = 0$

c. $p\,dy + q\,dz + r\,dx = 0$

d. $p\,dx + q\,dy + r\,dz = 0$

62. **Suppose $p\,dx + q\,dy + r\,dz = 0$ where p, q, r are homogeneous functions of x, y and z is a homogeneous differential equation, such that all of p, q and r are then to solve the differential equation, which of the following substitution is essential.**

a. $x = uz, y = vz$ b. $y = xz$

c. $x = y, z = 1$ d. $u = \frac{z}{x}, v = \frac{z}{y}$

63. **The auxiliary equation for the differential equation $p\,dx + q\,dy + r\,dz = 0$ is**

a. $\frac{dx}{\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}} = \frac{dy}{\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}} = \frac{dz}{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}$

b. $\frac{dp}{\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}} = \frac{dq}{\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}} = \frac{dr}{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}$

c. $dx = dy = dz$

d. none of the above

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Answers

1. a	2. b	3. a	4. b	5. c	6. b	7. b	8. a	9. b	10. b
11. c	12. c	13. c	14. b	15. c	16. b	17. c	18. b	19. c	20. b
21. a	22. b	23. a	24. c	25. b	26. d	27. c	28. a	29. b	30. a
31. c	32. c	33. a	34. b	35. b	36. c	37. d	38. b	39. b	40. a
41. a	42. b	43. c	44. d	45. b	46. c	47. d	48. c	49. b	50. b
51. a	52. c	53. d	54. a	55. b	56. c	57. d	58. d	59. b	60. d
61. d	62. a	63. a							

Hints/Solutions

31. Equation of the circles through the origin and having their centres on the x-axis is $(x-a)^2+(y-0)^2=a^2$

$\Rightarrow x^2+y^2-2ax=0$ where a is a parametr. Differentiating with respect to x

$2x+2y\dfrac{dy}{dx}-2a=0$ or, $2a=2x+2y\dfrac{dy}{dx}$

Eliminating 'a' between (i) and (ii), we get $x^2+y^2-(2x+2y\dfrac{dy}{dx})x=0$

$\Rightarrow y^2-x^2=2xy\dfrac{dy}{dx}$

32. $(x-h)^2=4a(y-k)$, where h, k a are three arbitrary constants. So $y_3=0$

33. Such conics are given by $ax^2+by^2=1$ where a and b are two arbitrary constants differentiating, $\dfrac{dy}{dx}=-\dfrac{ax}{by}$

$\therefore \dfrac{d^2y}{dx^2}=-\dfrac{a}{b}\left(\dfrac{1}{y}-\dfrac{x}{y^2}\dfrac{dy}{dx}\right)$

$\Rightarrow \dfrac{d^2y}{dx^2}=\dfrac{y}{x}\cdot\dfrac{dy}{dx}\left(\dfrac{1}{y}-\dfrac{x}{y^2}\cdot\dfrac{dy}{dx}\right)$

$\therefore xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2=y\dfrac{dy}{dx}$

Hence $xyy_2+xy_1^2=yy_1$

34. The given curve is $\dfrac{x^2}{a^2+P}+\dfrac{y^2}{b^2+P}=1$

Differentiating with respect to x we get

$\dfrac{2x}{a^2+P}+\dfrac{2yy_1}{b^2+P}=0$

or, $\dfrac{a^2+P}{x}=\dfrac{b^2+P}{-yy_1}=k$(say)

Putting this in the given equation, we get

$\dfrac{x^2}{kx}+\dfrac{y^2}{-kyy_1}=1\Rightarrow k=-\dfrac{1}{y_1}(y-xy_1)$

From (i) and (ii), we get

$a^2+P=kx=\dfrac{x^2y_1-yx}{y_1}$ ... (iii)

and $b^2+P=-kyy_1=y(y-xy_1)$ ... (iv)

subtracting (iv) from (iii) we get

$a^2-b^2=\dfrac{x^2y_1-yx}{y_1}$

35. $y^2=A(B-x)(B+x)$

$\therefore y\dfrac{dy}{dx}=-Ax$

$\Rightarrow y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=-A=\dfrac{y}{x}\dfrac{dy}{dx}$

$\therefore xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-y\dfrac{dy}{dx}=0$

36. $(1+y^2)dx-xydy=0\Rightarrow \dfrac{dx}{x}-\dfrac{ydy}{1+y^2}=0$

Integrating, $\log x-\dfrac{1}{2}\log(1+y^2)=\log c$

or, $\log\dfrac{x}{\sqrt{1+y^2}}=\log c \therefore x=c\sqrt{1+y^2}$

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If the curve passes through (1, 0), we have $c = 1$

∴ The equation of the required curve is $x^2 - y^2 = 1$

It is a rectangular hyperbola, and its foci are evidently $(\pm\sqrt{2}, 0)$

37. Since the length of the normal $= y\sqrt{1+y_1^2}$ and the radius vector $= \sqrt{x^2+y^2}$

$\therefore y^2(1 + y_1^2) = x^2 + y^2$

or, $y^2y_1^2 = x^2$ or, $yy_1 = \pm x$

$\therefore \frac{dy}{dx} = \pm\frac{x}{y}$ $\therefore xdx \pm ydy = 0$

Integrating, $x^2 \pm y^2 = a^2$, a^2 being the arbitrary constant of integration. Thus the curves are either circles or rectangular hyperbola.

38. Here $\phi = \pi - \frac{\theta}{2} \Rightarrow \tan\phi = \tan\left(\pi - \frac{\theta}{2}\right)$

$\Rightarrow r.\frac{d\theta}{dr} = -\tan\frac{\theta}{2} \Rightarrow \frac{dr}{r} + \cot\frac{\theta}{2}.d\theta = 0$

Integrating, $\log r + 2\log\sin\frac{\theta}{2} = \log c$

or, $r\sin^2\frac{\theta}{2} = c$

$\therefore \frac{r}{2}(1-\cos\theta) = c \Rightarrow \frac{2c}{r} = 1 - \cos\theta$

39. The given equation is $\frac{dy}{dx} + y\sec^2x = \tan x\sec^2x$ comparing with the linear equation in standard form, viz,

$\frac{dy}{dx} + P\cdot y = Q$

The integrating factor $= e^{\int Pdx} = e^{\int\sec^2 x dx}$ $= e^{\tan x}$

40. Here $x = \frac{y}{2P} - \frac{y^2P^2}{2}$

Differentiating w.r.to to y

$\frac{dx}{dy} = \frac{1}{P} = \frac{1}{2P} - \frac{y}{2P^2}\frac{dP}{dy} - yP^2 - Py^2\frac{dP}{dy}$

or, $1 + \frac{y}{P}\cdot\frac{dP}{dy} = 0$ or, $\frac{dP}{P} + \frac{dy}{y} = 0$

integrating, $\log y + \log P = \log c$

or, $Py = c$, where c is an arbitrary constant. Putting this value of P in the original equation, we get

$y = \frac{2cx}{y} + y^2\left(\frac{c}{y}\right)^3$

$\Rightarrow y^2 = 2cx + c^3$ as the general solution.

41. $P = \log(Px - y)$ or, $Px - y = e^P$

$\Rightarrow y = Px - e^P$

Differentiating w.r.to x

$\frac{dy}{dx} = P = P + x\frac{dP}{dx} - e^P\frac{dP}{dx}$

or, $\frac{dP}{dx}(x - e^P) = 0$

either, $\frac{dP}{dx} = 0$ or $x - e^P = 0$

if, $\frac{dP}{dx} = 0$, $P = c$ where c is any constant

So, $y = cx - e^c$ is the general solution of (i).

Again, $x - e^P = 0$ gives $P = \log x$

From (i), $y = x\log x - x \Rightarrow y + x\log x$ is the singular solution.

42. Family of hyperbolas is $xy = c^2$

$\therefore y\cdot 1 + x\frac{dy}{dx} = 0$

Replacing $\frac{dy}{dx}$ by $\frac{dx}{dy}$ the differential equation of orthogonal trajectories is

$y - x\frac{dx}{dy} = 0 \Rightarrow x\,dx - ydy = 0$

Integrating, $x^2 - y^2 = c$

This gives family of orthogonal trajectories of the hyperbolas $xy = c^2$

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